On Classical Electromagnetic Fields

I. Preliminaries: A Review of Some Basic Concepts and Methods:

The Microscopic Maxwell's Equations in the Time Domain:

$$\vec{\nabla}\times\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)=-\frac{\partial}{\partial t}\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-1a]}$$

$$\vec{\nabla}\times\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)=\mu_0\,\vec{\mathbf{J}}\left(\vec{\mathbf{r}},t\right)+\varepsilon_0\,\mu_0\frac{\partial}{\partial t}\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-1b]}$$

$$\vec{\nabla}\cdot\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)=\frac{1}{\varepsilon_0}\rho\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-1c]}$$

$$\vec{\nabla}\cdot\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-1d]}$$

where the meaning of fields $\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)$ and $\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)$ is ultimately defined in terms of the Lorentz force on a charge q moving at a velocity $\vec{\mathbf{u}}$ -- *viz.*,

$$\vec{\mathbf{F}}\left(\vec{\mathbf{r}},t\right)=q\,\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)+q\,\vec{\mathbf{u}}\times\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-2]}$$

The Macroscopic Maxwell's Equations in the Time Domain

Following common practice, we set forth a particular set of macroscopic Maxwell's equations that applies in the high frequency or ***optical regime***.[1]

$$\vec{\nabla}\times\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)=-\frac{\partial}{\partial t}\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)=-\mu_0\frac{\partial}{\partial t}\vec{\mathbf{H}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-3a]}$$

$$\vec{\nabla}\times\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)=\mu_0\,\vec{\nabla}\times\vec{\mathbf{H}}\left(\vec{\mathbf{r}},t\right)=\mu_0\left[\vec{\mathbf{J}}\left(\vec{\mathbf{r}},t\right)+\frac{\partial}{\partial t}\vec{\mathbf{P}}\left(\vec{\mathbf{r}},t\right)\right]+\varepsilon_0\mu_0\frac{\partial}{\partial t}\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-3b]}$$

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[1] All bound current effects are included in the polarization density, since magnetization density "ceases to have any physical meaning at relatively low frequencies." See Section 62 in L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press (1960).

$$\vec{\nabla}\cdot\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)=\frac{1}{\varepsilon_0}\left[\rho\left(\vec{\mathbf{r}},t\right)-\vec{\nabla}\cdot\vec{\mathbf{P}}\left(\vec{\mathbf{r}},t\right)\right] \qquad \text{[I-3c]}$$

$$\vec{\nabla}\cdot\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)=\mu_0\,\vec{\nabla}\cdot\vec{\mathbf{H}}\left(\vec{\mathbf{r}},t\right)=0 \qquad \text{[I-3d]}$$

<u>A PHENOMENOLOICAL REPRESENTATION OF THE LINEAR DIELECTRIC RESPONSE OF MATTER:</u>[2]

The following is the most general phenomenological representation of the linear dielectric response of a given material that incorporates **dissipative**, **non-local**, and **anisotropic** effects:

In a ***tensor representation***

$$\mathrm{P}_\alpha^{(\mathrm{L})}\left(\vec{\mathbf{r}},t\right)=\varepsilon_0\int_{-\infty}^{t}dt'\iiint\chi_{\alpha\beta}\left(\vec{\mathbf{r}},\vec{\mathbf{r}}';\,t,t'\right)\,\mathrm{E}_\beta\left(\vec{\mathbf{r}},t'\right)\,d\vec{\mathbf{r}}' \qquad \text{[I-4a]}$$

In a ***dyadic representation***

$$\vec{\mathbf{P}}^{(\mathrm{L})}\left(\vec{\mathbf{r}},\,t\right)=\varepsilon_0\int_{-\infty}^{t}dt'\iiint\vec{\vec{\chi}}\left(\vec{\mathbf{r}},\vec{\mathbf{r}}';\,t,t'\right)\,\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\,t'\right)\,d\vec{\mathbf{r}}' \qquad \text{[I-4b]}$$

For the present and for most of our discussions, we neglect **nonlocal** effects and treat only **dispersive** (dissipative) and **anisotropic** effects so that

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2 In our later treatment of nonlinear optics, we will begin by adding the following nonlinear phenomenological contributions:

$$\begin{aligned}\mathbf{P}_\alpha^{(\mathrm{NL})}\left(\vec{\mathbf{r}},t\right)=\;&\varepsilon_0\sum_{\beta\gamma}\int_{\vec{\mathbf{r}}_1}\int_{\vec{\mathbf{r}}_2}\int_{t_1}\int_{t_2}d\vec{\mathbf{r}}_1dt_1d\vec{\mathbf{r}}_2dt_2\;\chi_{\alpha\beta\gamma}^{(2)}\left(\vec{\mathbf{r}}-\vec{\mathbf{r}}_1,\;t-t_1;\vec{\mathbf{r}}-\vec{\mathbf{r}}_2,\;t-t_2\right)\mathbf{E}_\beta\left(\vec{\mathbf{r}}_1,\;t_1\right)\mathbf{E}_\gamma\left(\vec{\mathbf{r}}_2,\;t_2\right)\\&+\varepsilon_0\sum_{\beta\gamma\delta}\int_{\vec{\mathbf{r}}_1}\int_{\vec{\mathbf{r}}_2}\int_{\vec{\mathbf{r}}_3}\int_{t_1}\int_{t_2}\int_{t_3}d\vec{\mathbf{r}}_1dt_1d\vec{\mathbf{r}}_2dt_2d\vec{\mathbf{r}}_3dt_3\;\chi_{\alpha\beta\gamma\delta}^{(3)}\left(\vec{\mathbf{r}}-\vec{\mathbf{r}}_1,\;t-t_1;\vec{\mathbf{r}}-\vec{\mathbf{r}}_2,\;t-t_2;\vec{\mathbf{r}}-\vec{\mathbf{r}}_3,\;t-t_3\right)\\&\times\mathbf{E}_\beta\left(\vec{\mathbf{r}}_1,\;t_1\right)\mathbf{E}_\gamma\left(\vec{\mathbf{r}}_2,\;t_2\right)\mathbf{E}_\delta\left(\vec{\mathbf{r}}_3,\;t_3\right)+\cdots.\end{aligned}$$

$$\vec{\mathbf{P}}\left(\vec{\mathbf{r}}, t\right) = \varepsilon_0 \int_{-\infty}^{t} dt' \; \vec{\vec{\chi}}\left(\vec{\mathbf{r}}, \, t - t'\right) \; \vec{\mathbf{E}}\left(\vec{\mathbf{r}}, t'\right) \qquad \text{[I-5]}$$

or

$$\vec{\mathbf{P}}\left(\vec{\mathbf{r}}, \omega\right) = \varepsilon_0 \, \vec{\vec{\chi}}\left(\vec{\mathbf{r}}, \omega\right) \; \vec{\mathbf{E}}\left(\vec{\mathbf{r}}, \omega\right) \qquad \text{[I-6]}$$

where

$$\vec{\vec{\chi}}\left(\vec{\mathbf{r}}, \omega\right) = \int_{-\infty}^{t} \vec{\vec{\chi}}\left(\vec{\mathbf{r}}, t'\right) \exp\left[-i\,\omega\left(t - t'\right)\right] dt' = \int_{0}^{\infty} \vec{\vec{\chi}}\left(\vec{\mathbf{r}}, \tau\right) \exp\left[-i\omega\tau\right] d\tau \qquad \text{[I-7]}$$

Macroscopic Maxwell's Equations in the Frequency Domain Valid for Linear, Local, Anisotropic Media in the *Optical Regime*.

$$\vec{\nabla} \times \vec{\mathbf{E}}\left(\vec{\mathbf{r}}, \omega\right) = \vec{\nabla} \times \left[\vec{\vec{\varepsilon}}\right]^{-1}\left(\vec{\mathbf{r}}, \omega\right) \; \vec{\mathbf{D}}\left(\vec{\mathbf{r}}, \omega\right) = -i\omega \vec{\mathbf{B}}\left(\vec{\mathbf{r}}, \omega\right) = -i\omega \mu_0 \vec{\mathbf{H}}\left(\vec{\mathbf{r}}, \omega\right) \qquad \text{[I-8a]}$$

$$\begin{aligned} \vec{\nabla} \times \vec{\mathbf{B}}\left(\vec{\mathbf{r}}, \omega\right) = \mu_0 \vec{\nabla} \times \vec{\mathbf{H}}\left(\vec{\mathbf{r}}, \omega\right) &= \mu_0 \vec{\mathbf{J}}\left(\vec{\mathbf{r}}, \omega\right) + i\omega \mu_0 \vec{\vec{\varepsilon}}\left(\vec{\mathbf{r}}, \omega\right) \; \vec{\mathbf{E}}\left(\vec{\mathbf{r}}, \omega\right) \\ &= \mu_0 \vec{\mathbf{J}}\left(\vec{\mathbf{r}}, \omega\right) + i\omega \mu_0 \vec{\mathbf{D}}\left(\vec{\mathbf{r}}, \omega\right) \end{aligned} \qquad \text{[I-8b]}$$

$$\vec{\nabla} \cdot \vec{\vec{\varepsilon}}\left(\vec{\mathbf{r}}, \omega\right) \cdot \vec{\mathbf{E}}\left(\vec{\mathbf{r}}, \omega\right) = \vec{\nabla} \cdot \vec{\mathbf{D}}\left(\vec{\mathbf{r}}, \omega\right) = \rho\left(\vec{\mathbf{r}}, \omega\right) \qquad \text{[I-8c]}$$

$$\vec{\nabla} \cdot \vec{\mathbf{B}}\left(\vec{\mathbf{r}}, \omega\right) = \mu_0 \vec{\nabla} \cdot \vec{\mathbf{H}}\left(\vec{\mathbf{r}}, \omega\right) = 0 \qquad \text{[I-8d]}$$

HELMHOLZ EQUATIONS FOR THE FREQUENCY DOMAIN VECTOR AND SCALAR POTENTIALS IN A UNIFORM, LINEAR, ISOTROPIC DIELECTRIC

We define the (Magnetic) Vector Potential as

$$\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=\frac{1}{\mu_0}\vec{\mathbf{B}}(\vec{\mathbf{r}},\omega)=\frac{1}{\mu_0}\vec{\nabla}\times\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-9]}$$

which, by design, automatically satisfies one of Maxwell's equations -- *viz.*, [I-8d].[3] . We introduce the (Electric) Scalar Potential in the form

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=-i\omega\,\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)-\vec{\nabla}\varphi(\vec{\mathbf{r}},\omega) \qquad \text{[I-10]}$$

which, again, automatically satisfies another Maxwell equation -- *viz.*, [I-8a].[4] Therefore, for **uniform, isotropic media** Equation [I-8b] becomes [5]

$$\begin{aligned}\vec{\nabla}\times\vec{\nabla}\times\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)&=\mu_0\,\vec{\mathbf{J}}(\vec{\mathbf{r}},\omega)+\omega^2\mu_0\varepsilon(\omega)\,\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)-i\omega\mu_0\varepsilon(\omega)\,\vec{\nabla}\varphi(\vec{\mathbf{r}},\omega)\\&=\vec{\nabla}\,\vec{\nabla}\cdot\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)-\nabla^2\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)\end{aligned} \qquad \text{[I-11a]}$$

and Equation [I-8c] becomes

$$-i\omega\,\varepsilon(\omega)\,\vec{\nabla}\cdot\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)-\varepsilon(\omega)\,\nabla^2\varphi(\vec{\mathbf{r}},\omega)=\rho(\vec{\mathbf{r}},\omega). \qquad \text{[I-11b]}$$

Since $\vec{\nabla}\cdot\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)$ is as yet undefined, we define it in the ***Lorentz gauge*** as

$$\vec{\nabla}\cdot\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)=-i\omega\mu_0\,\varepsilon(\omega)\,\varphi(\vec{\mathbf{r}},\omega) \qquad \text{[I-12]}$$

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3 Since $\mathit{div\ curl}\{\quad\}=\vec{\nabla}\cdot\vec{\nabla}\times\{\quad\}=0$.

4 Since $\mathit{curl\ grad}\{\quad\}=\vec{\nabla}\times\vec{\nabla}\{\quad\}=0$.

5 Using $\vec{\mathbf{a}}\times(\vec{\mathbf{b}}\times\vec{\mathbf{c}})=\vec{\mathbf{b}}(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})-\vec{\mathbf{c}}(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})$.

to simplify Equations [I-11a] and [I-11b]. Thus

$$ {}^{2}\,\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)+\omega^{2}\mu_{0}\,\varepsilon\left(\omega\right)\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)=-\mu_{0}\,\vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[I-13a]} $$

and

$$ {}^{2}\varphi\left(\vec{\mathbf{r}},\omega\right)+\omega^{2}\mu_{0}\,\varepsilon\left(\omega\right)\varphi\left(\vec{\mathbf{r}},\omega\right)=-\frac{1}{\varepsilon\left(\omega\right)}\rho\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[I-13b]} $$

Therefore, in the Lorentz gauge **both** $\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)$ and $\varphi\left(\vec{\mathbf{r}},\omega\right)$ **satisfy inhomogeneous (**and homogeneous) **Helmholz equations!**

However, in the ***Coulomb gauge*** we define $\vec{\ }\ \vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)=0$ and then Equation [I-8c] becomes

$$ {}^{2}\varphi\left(\vec{\mathbf{r}},\omega\right)=-\frac{1}{\varepsilon\left(\omega\right)}\rho\left(\vec{\mathbf{r}},\omega\right). \qquad \text{[I-14a]} $$

Conservation of charge requires that $\vec{\ }\ \vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right)-i\omega\rho\left(\vec{\mathbf{r}},\omega\right)=0$ so that Equation [I-11a] becomes

$$ {}^{2}\,\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)+\omega^{2}\mu_{0}\,\varepsilon\left(\omega\right)\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)=-\mu_{0}\,\vec{\mathbf{J}}^{(\mathrm{trans})}\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[I-14b]} $$

where

$$ \begin{aligned} \vec{\mathbf{J}}^{(\mathrm{trans})}\left(\vec{\mathbf{r}},\omega\right) &= \vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right)-\vec{\mathbf{J}}^{(\mathrm{long})}\left(\vec{\mathbf{r}},\omega\right) \\ &= \vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right)-\left[-i\omega\varepsilon\left(\omega\right)\quad\varphi\left(\vec{\mathbf{r}},\omega\right)\right] \end{aligned} \qquad \text{[I-15]} $$

II. RAYS: THE EIKONAL TREATMENT OF GEOMETRIC OPTICS [6]

Since ancient times, the notion of **ray or beam propagation** has been one of the most enduring and fundamental concepts in optical physics. As a zeroth order approximation we might consider a plane wave to be a model of a **beam** and its propagation vector to be a model of a **ray**. This is a reasonable start, but it is a much too restricted view and we can do much better. What we need is a solution to Maxwell's equations which is like a plane wave, but limited in spatial extent. One approach, the simplest, is called variously ray, Gaussian or geometric optics.

A MAXWELLIAN DERIVATION OF THE EIKONAL EQUATION:

To fully understand geometric optics in the context of Maxwell's equations, we start by writing the electric and magnetic fields as ***pseudo-simple*** waves -- *viz.*

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\mathbf{e}}(\vec{\mathbf{r}},\omega)\, \exp\left[-i\,\mathrm{k}_0\, \mathrm{S}(\vec{\mathbf{r}},\omega)\right] \qquad \text{[II-1a]}$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) = \vec{\mathbf{h}}(\vec{\mathbf{r}},\omega)\, \exp\left[-i\,\mathrm{k}_0 \mathrm{S}(\vec{\mathbf{r}},\omega)\right] \qquad \text{[II-1b]}$$

where $\mathrm{k}_0 = \omega\sqrt{\mu_0\varepsilon_0} = \omega / c$

It is assumed that $\vec{\mathbf{e}}(\vec{\mathbf{r}},\omega)$ and $\vec{\mathbf{h}}(\vec{\mathbf{r}},\omega)$ are **weak functions of position**. The scalar phase function $\mathrm{S}(\vec{\mathbf{r}},\omega)$ is the spatially varying phase of the *pseudo-simple* wave. For the cases of pseudo-plane waves and pseudo-spherical waves the phase function is given, respectively, by

$$\mathrm{k}_0\mathrm{S}(\vec{\mathbf{r}},\omega) = x\,\mathrm{k}_x + y\,\mathrm{k}_y + z\,\mathrm{k}_z \qquad \text{[II-2a]}$$

6 See, for example, Max Born and Emil Wolf, *Principle of Optics*, Pergamon Press (1986), Chapter 3.

and $$\mathrm{k}_0 \mathrm{S}\left(\vec{\mathbf{r}},\omega\right) = \mathrm{k}_0 \sqrt{x^2 + y^2 + z^2} \qquad \text{[II-2b]}$$

We now substitute these pseudo-simple wave expressions (*i.e.* Equations [II-1]) into Maxwell's equations to obtain

$$\exp\left[-i\,\mathrm{k}_0 \mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\right]\left\{\vec{\nabla}\times\vec{\mathbf{e}}\left(\vec{\mathbf{r}},\omega\right) - i\,\mathrm{k}_0\,\vec{\nabla}\mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\times\vec{\mathbf{e}}\left(\vec{\mathbf{r}},\omega\right)\right\} = -i\mu_0\,\mathrm{ck}_0\,\vec{\mathbf{h}}\left(\vec{\mathbf{r}},\omega\right)\exp\left[-i\,\mathrm{k}_0 \mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\right] \qquad [$$

$$\exp\left[-i\,\mathrm{k}_0 \mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\right]\left\{\vec{\nabla}\times\vec{\mathbf{h}}\left(\vec{\mathbf{r}},\omega\right) - i\,\mathrm{k}_0\,\vec{\nabla}\mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\times\vec{\mathbf{h}}\left(\vec{\mathbf{r}},\omega\right)\right\} = i\varepsilon\left(\vec{\mathbf{r}},\omega\right)\mathrm{ck}_0\,\vec{\mathbf{e}}\left(\vec{\mathbf{r}},\omega\right)\exp\left[-i\,\mathrm{k}_0 \mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\right] \qquad [$$

$$\exp\left[-i\,\mathrm{k}_0 \mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\right]\left\{\vec{\nabla}\cdot\left[\varepsilon\left(\vec{\mathbf{r}},\omega\right)\vec{\mathbf{e}}\left(\vec{\mathbf{r}},\omega\right)\right] - i\,\mathrm{k}_0\,\varepsilon\left(\vec{\mathbf{r}},\omega\right)\vec{\nabla}\mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\cdot\vec{\mathbf{e}}\left(\vec{\mathbf{r}},\omega\right)\right\} = 0 \qquad [$$

$$\exp\left[-i\,\mathrm{k}_0 \mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\right]\left\{\vec{\nabla}\cdot\vec{\mathbf{h}}\left(\vec{\mathbf{r}},\omega\right) - i\,\mathrm{k}_0\,\vec{\nabla}\mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\cdot\vec{\mathbf{h}}\left(\vec{\mathbf{r}},\omega\right)\right\} = 0 \qquad [$$

Rearranging, we obtain

$$\vec{\nabla}\mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\times\vec{\mathbf{e}}\left(\vec{\mathbf{r}},\omega\right) - \mu_0\,\mathrm{c}\,\vec{\mathbf{h}}\left(\vec{\mathbf{r}},\omega\right) = \left[i\,\mathrm{k}_0\right]^{-1}\vec{\nabla}\times\vec{\mathbf{e}}\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[II-4a]}$$

$$\vec{\nabla}\mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\times\vec{\mathbf{h}}\left(\vec{\mathbf{r}},\omega\right) + \varepsilon\left(\vec{\mathbf{r}},\omega\right)\mathrm{c}\,\vec{\mathbf{e}}\left(\vec{\mathbf{r}},\omega\right) = \left[i\,\mathrm{k}_0\right]^{-1}\vec{\nabla}\times\vec{\mathbf{h}}\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[II-4b]}$$

$$\vec{\nabla}\mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\cdot\left[\varepsilon\left(\vec{\mathbf{r}},\omega\right)\vec{\mathbf{e}}\left(\vec{\mathbf{r}},\omega\right)\right] = \left[i\,\mathrm{k}_0\right]^{-1}\vec{\nabla}\cdot\left[\varepsilon\left(\vec{\mathbf{r}},\omega\right)\vec{\mathbf{e}}\left(\vec{\mathbf{r}},\omega\right)\right] \qquad \text{[II-4c]}$$

$$\vec{\nabla}\mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\cdot\vec{\mathbf{h}}\left(\vec{\mathbf{r}},\omega\right) = \left[i\,\mathrm{k}_0\right]^{-1}\vec{\nabla}\cdot\vec{\mathbf{h}}\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[II-4d]}$$

In the **ray**, **Gaussian** or **geometric approximation** we assume that we may neglect the RHS's of these equations. To get something useful we multiply through the first equation (*i.e.* Equation [II-4a]) as follows:

$$\left[\mu_0\,c\right]^{-1}\vec{\nabla}\mathrm{S}\left(\vec{\mathbf{r}},\omega\right)\times\{\ \text{Equation [II-4a]}\ \} \qquad \text{[II-5a]}$$

$$[\mu_0 c]^{-1}\vec{\nabla}S(\vec{\mathbf{r}},\omega)\times\left\{\vec{\nabla}S(\vec{\mathbf{r}},\omega)\times\vec{\mathbf{e}}(\vec{\mathbf{r}},\omega)-\mu_0 c\,\vec{\mathbf{h}}(\vec{\mathbf{r}},\omega)\right\}\approx 0 \qquad \text{[II-5b]}$$

Applying the "abc = bac - cab" rule[7] we obtain

$$\begin{aligned}[\mu_0 c]^{-1}\Big\{\vec{\nabla}S(\vec{\mathbf{r}},\omega)\left[\vec{\nabla}S(\vec{\mathbf{r}},\omega)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},\omega)\right]&\\ -\vec{\mathbf{e}}(\vec{\mathbf{r}},\omega)\left|\vec{\nabla}S(\vec{\mathbf{r}},\omega)\right|^2-\mu_0 c\,\vec{\nabla}S(\vec{\mathbf{r}},\omega)\times\vec{\mathbf{h}}(\vec{\mathbf{r}},\omega)\Big\}&\approx 0\end{aligned} \qquad \text{[II-5c]}$$

which becomes upon substitution from the second Equation [II-4b]

$$\begin{aligned}[\mu_0 c]^{-1}\Big\{\vec{\nabla}S(\vec{\mathbf{r}},\omega)\left[\vec{\nabla}S(\vec{\mathbf{r}},\omega)\cdot\vec{\mathbf{e}}(\vec{\mathbf{r}},\omega)\right]&\\ -\vec{\mathbf{e}}(\vec{\mathbf{r}},\omega)\left|\vec{\nabla}S(\vec{\mathbf{r}},\omega)\right|^2\Big\}+\varepsilon(\vec{\mathbf{r}},\omega)\,c\,\vec{\mathbf{e}}(\vec{\mathbf{r}},\omega)&\approx 0\end{aligned} \qquad \text{[II-5d]}$$

From Equation [II-4c], we see that the first term vanishes in the geometric approximation -- *i.e.*, if we neglect the term $[i\,k_0]^{-1}\vec{\nabla}\cdot\left[\varepsilon(\vec{\mathbf{r}},\omega)\,\vec{\mathbf{e}}(\vec{\mathbf{r}},\omega)\right]$. Therefore, for non-vanishing $\vec{\mathbf{e}}(\vec{\mathbf{r}},\omega)$ we obtain the following ***reduction* of Maxwell's equations**:

$$\left|\vec{\nabla}S(\vec{\mathbf{r}},\omega)\right|^2\approx\varepsilon(\vec{\mathbf{r}},\omega)\mu_0 c^2=n^2(\vec{\mathbf{r}},\omega) \qquad \text{[II-6]}$$

where $n(\vec{\mathbf{r}},\omega)$ is the index of refraction. More explicitly, we may write an equation for a "ray vector" -- *i.e.* the tangent to a space curve orthogonal to the surfaces of constant $S(\vec{\mathbf{r}},\omega)$

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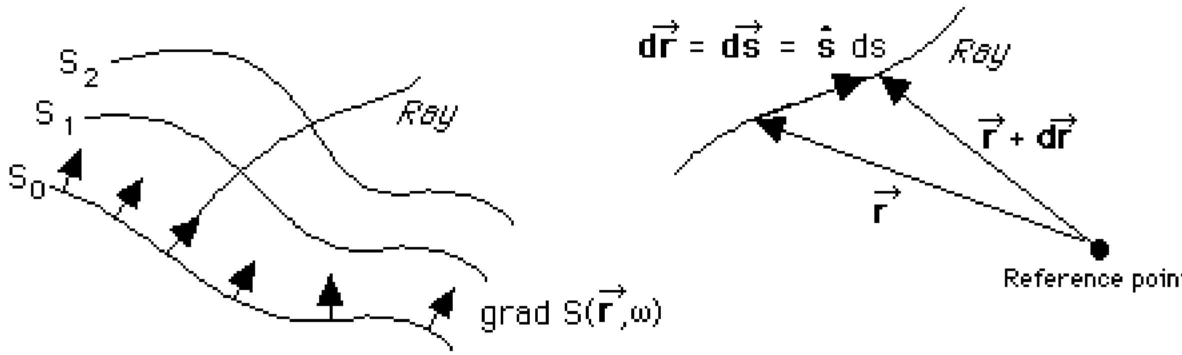
7 Again using $\vec{\mathbf{a}}\times(\vec{\mathbf{b}}\times\vec{\mathbf{c}})=\vec{\mathbf{b}}(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})-\vec{\mathbf{c}}(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})$.

$$\nabla S(\vec{\mathbf{r}},\omega) = n(\vec{\mathbf{r}},\omega)\;\hat{\mathbf{s}} = n(\vec{\mathbf{r}},\omega)\;\frac{d\vec{\mathbf{r}}}{ds} \qquad [\text{ II-7 }]$$

We illustrate the geometric relationships below:



We may now derive the all important **eikonal equation.** To that end, we first take a derivative along the ray direction -- *viz*.

$$\frac{d}{ds}\left[\nabla S(\vec{\mathbf{r}},\omega)\right] = \frac{d}{ds}\left[n(\vec{\mathbf{r}},\omega)\;\frac{d\vec{\mathbf{r}}}{ds}\right] \qquad [\text{ II-8a }]$$

However, from the definition of the **grad** operator we know that

$$d\left[\nabla S(\vec{\mathbf{r}},\omega)\right] = d\vec{\mathbf{r}}\cdot\nabla\left[\nabla S(\vec{\mathbf{r}},\omega)\right]$$

so that

$$\frac{d\vec{\mathbf{r}}}{ds}\cdot\nabla\left[\nabla S(\vec{\mathbf{r}},\omega)\right] = \left\{\frac{1}{n(\vec{\mathbf{r}},\omega)}\nabla S(\vec{\mathbf{r}},\omega)\right\}\cdot\nabla\left[\nabla S(\vec{\mathbf{r}},\omega)\right] = \frac{d}{ds}\left[n(\vec{\mathbf{r}},\omega)\;\frac{d\vec{\mathbf{r}}}{ds}\right] \qquad [\text{ II-8b }]$$

or

$$\frac{1}{2\,n(\vec{\mathbf{r}},\omega)}\vec{\nabla}\left\{\vec{\nabla}S(\vec{\mathbf{r}},\omega)\cdot\vec{\nabla}S(\vec{\mathbf{r}},\omega)\right\} = \frac{1}{2\,n(\vec{\mathbf{r}},\omega)}\vec{\nabla}\left\{n^2(\vec{\mathbf{r}},\omega)\right\} = \vec{\nabla}\,n(\vec{\mathbf{r}},\omega) = \frac{d}{ds}\left[n(\vec{\mathbf{r}},\omega)\,\frac{d\vec{\mathbf{r}}}{ds}\right] \qquad \text{[II-8c]}$$

Thus we have obtain the ***eikonal*** [8] equation for the ray vector -- *viz.*

$$\frac{d}{ds}\left[n(\vec{\mathbf{r}},\omega)\,\frac{d\vec{\mathbf{r}}}{ds}\right] = \vec{\nabla}\,n(\vec{\mathbf{r}},\omega) \qquad \text{[II-9]}$$

FIRST APPLICATION OF THE EIKONAL EQUATION: MIRAGES

Air adjacent to a hot surface rises in temperature and becomes less dense. Thus over a flat hot surface, such as a desert expanse or a sun drenched roadway, air density **locally** increases with height and the average **refractive index** may be approximated by a simple linear variation of the form

$$n(x) = n_g\,\{1 + \kappa\,x\} \qquad \text{[II-10]}$$

where x is the vertical height above the planar surface, n_g is the refractive index at ground level, and κ is a positive constant.

We may use the **eikonal equation** to find an equation for the approximate ray trajectory -- *i.e.* an equation for ray height x as a function of ground distance z -- of a light ray launched from a height x_o and at an angle θ_o with respect to the surface of the earth.

8 The *eikonal* (from the Greek: εικϖν means *image*) was introduced in 1895 by H. Bruns.

Therefore,

$$\frac{d}{ds}\left[n(\vec{\mathbf{r}},\omega)\,\frac{d\vec{\mathbf{r}}}{ds}\right] = \vec{\nabla}\, n(\vec{\mathbf{r}},\omega) \qquad \frac{d^2 x}{dz^2} = \frac{1}{n(x,\omega)}\,\frac{d}{dx}\, n(x,\omega) \qquad \text{[II-11a]}$$

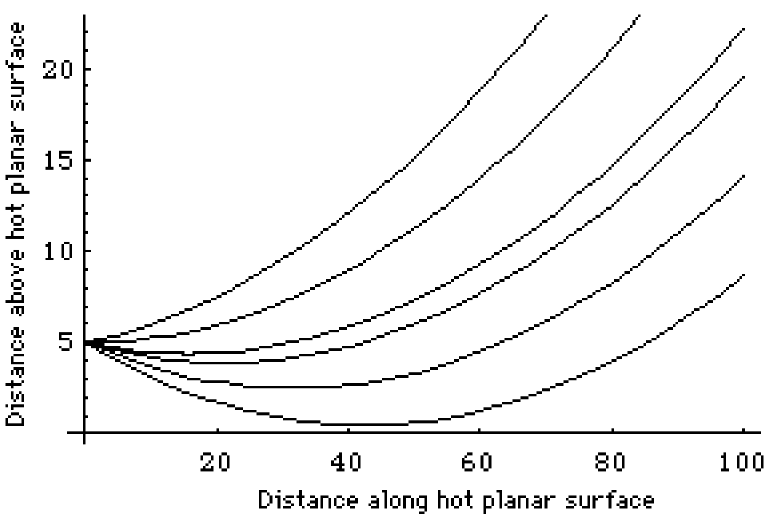
or from Equation [II-10]

$$\frac{d^2 x}{dz^2} \approx \kappa\ . \qquad \text{[II-11b]}$$

Thus, the ray trajectory is given by

$$\vec{\mathbf{r}}(z) = \left[\frac{\kappa}{2}\, z^2 + \tan\theta_0\, z + x_0\right]\hat{\mathbf{x}} + z\,\hat{\mathbf{z}} \qquad \text{[II-12]}$$

Ray trajectories diverted by a hot surface



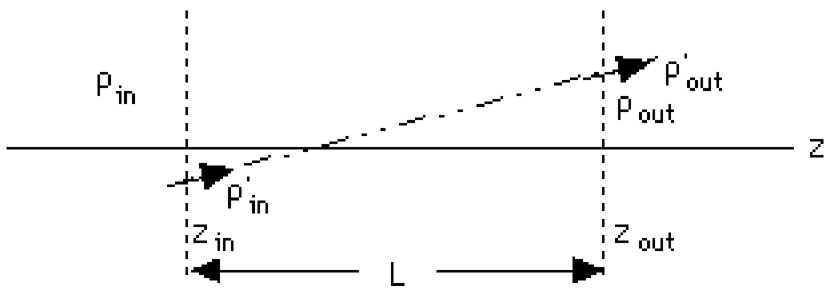
SECOND APPLICATION OF THE EIKONAL EQUATION: THE "ABCD" RAY MATRICES - A SYSTEMS APPROACH TO OPTICS

1. <u>Uniform dielectric medium</u> -- *i.e.* $n(\vec{\mathbf{r}},\omega)$ is a constant so that $\frac{d}{ds}\left[\frac{d\vec{\mathbf{r}}}{ds}\right] = 0$.

   Thus, the ray **must be a straight line** which may be written $\vec{\mathbf{r}} = s\,\vec{\mathbf{a}} + \vec{\mathbf{b}}$.
   In the two-dimensional paraxial approximation, we assume that $s \approx z$ and write

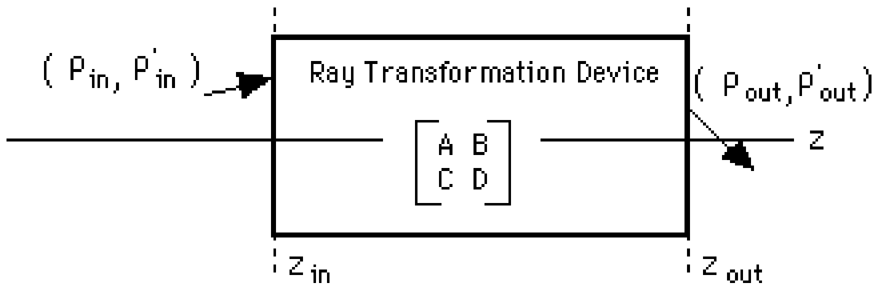
$$\rho_{out} = \rho_{in} + L\left.\frac{d\rho}{dz}\right|_{in} \quad \text{and} \quad \left.\frac{d\rho}{dz}\right|_{out} = \left.\frac{d\rho}{dz}\right|_{in} \qquad [\text{ II-13 }]$$

   where $\vec{\rho} = \rho\,\hat{\rho} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$.



We may write results of this sort in the form of the famous and highly useful **ray transform** or **ABCD matrix** -- *viz.*

$$\begin{bmatrix}\rho_{out}\\ \rho'_{out}\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\rho_{in}\\ \rho'_{in}\end{bmatrix} \qquad [\text{ II-14 }]$$



In the case of a uniform dielectric

$$\begin{bmatrix} \rho_{\text{out}} \\ \rho'_{\text{out}} \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_{\text{in}} \\ \rho'_{\text{in}} \end{bmatrix} \qquad \text{[II-15]}$$

so that A = 1, B = L, C = 0, and D = 1

2. A dielectric discontinuity: Starting with Equation [II-7] and noting, once again, that *curl grad* $\{\qquad\} = \vec{\nabla} \times \vec{\nabla} \{\qquad\} = 0$ we see that

$$\vec{\nabla} \times \vec{\nabla} S(\vec{\mathbf{r}},\omega) = \vec{\nabla} \times \left\{ n(\vec{\mathbf{r}},\omega)\, \hat{\mathbf{s}} \right\} = 0 \qquad \text{[II-16]}$$

which is identical to the *saltus* condition on the electric and magnetic fields at a dielectric interface! Hence $\left\{ n(\vec{\mathbf{r}},\omega)\, \hat{\mathbf{s}} \right\}_{\text{tangent}}$ is continuous across the dielectric boundary so that $n_1 \sin\theta_1 = n_1 \sin\theta_1$ -- *i.e.* Snell's law! This result in the paraxial approximation (*i.e.*, $\sin\theta \approx \tan\theta \approx \theta$) may be written in ray matrix form as

$$\begin{bmatrix} \rho_{\text{out}} \\ \rho'_{\text{out}} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \rho_{\text{in}} \\ \rho'_{\text{in}} \end{bmatrix} \approx \begin{bmatrix} 1 & 0 \\ 0 & n_{\text{in}}/n_{\text{out}} \end{bmatrix} \begin{bmatrix} \rho_{\text{in}} \\ \rho'_{\text{in}} \end{bmatrix} \qquad \text{[II-17]}$$

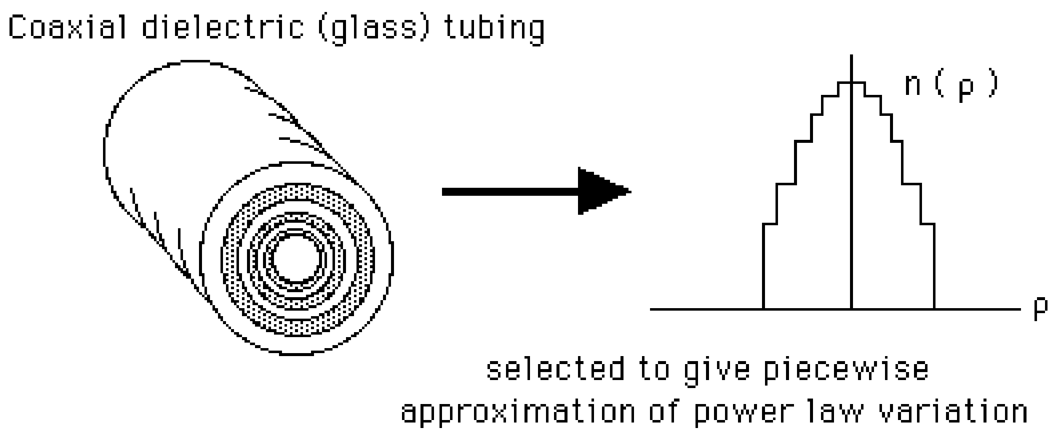
3. A "Thin" lenses: In passing we note that the ray matrix of a thin lens is given by or, perhaps more accurately, a thin lens is essentially defined by

$$\begin{bmatrix}\rho_{\text{out}}\\ \rho'_{\text{out}}\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\rho_{\text{in}}\\ \rho'_{\text{in}}\end{bmatrix} \approx \begin{bmatrix}1 & 0\\ -f^{-1} & 1\end{bmatrix}\begin{bmatrix}\rho_{\text{in}}\\ \rho'_{\text{in}}\end{bmatrix} \qquad \text{[II-18]}$$

4. Axially symmetric GRIN media: Consider the use of GRaded INdex technology to obtain an axially symmetric variation in the index of refraction of the form [9]

$$n(\rho) = n_{\text{M}}\left[1 - \Delta\left(\frac{\rho}{a}\right)^{m}\right] \qquad \text{[II-19]}$$

9 A note on GRIN technology: In GRIN technology one builds up a glass rod with a specific radial index of refraction distribution by fusing a sequence of coaxially arranged glass tubes with appropriate index and diameter as illustrated in the following:



Within such a GRIN rod, we write $\vec{\mathbf{r}} = \rho\hat{\rho} + z\,\hat{\mathbf{z}}$ for the ray coordinates and $\vec{\nabla} n(\vec{\mathbf{r}}) = \hat{\rho}\,\frac{d}{d\rho}\,n(\rho)$ for the index variation. Using the eikonal equation -- *i.e.* Equation [II-9] -- in the paraxial approximation, we find

$$\vec{\nabla} n(\rho) = \frac{d}{ds}\left[n(\rho)\,\frac{d\vec{\mathbf{r}}}{ds}\right] \approx \frac{d}{dz}\left[n(\rho)\,\frac{d\vec{\mathbf{r}}}{dz}\right] = \frac{d}{dz}\left\{n(\rho)\left[\frac{d\rho}{dz}\,\hat{\rho} + \hat{\mathbf{z}}\right]\right\} \qquad \text{[II-20a]}$$

or

$$\frac{d}{d\rho}\,n(\rho) \approx \frac{d}{dz}\left[n(\rho)\,\frac{d\rho}{dz}\right] \qquad \text{[II-20b]}$$

Therefore

$$\frac{d^2}{dz^2}\,\rho \approx \frac{1}{n(\rho)}\,\frac{d}{d\rho}\,n(\rho) = \frac{d}{d\rho}\,\ln\left[n(\rho)\right] \qquad \text{[II-20c]}$$

or

$$\begin{aligned}\frac{d^2}{dz^2}\,\rho &\approx \frac{1}{n(\rho)}\,\frac{d}{d\rho}\left\{n_M\left[1-\Delta\left(\frac{\rho}{a}\right)^m\right]\right\} = \frac{1}{n(\rho)}\left[-n_M\,\frac{m\Delta}{a}\left(\frac{\rho}{a}\right)^{m-1}\right] \\ &\approx -\frac{m\Delta}{a}\left(\frac{\rho}{a}\right)^{m-1}\end{aligned} \qquad \text{[II-20d]}$$

Doubtless, the simplest and most valuable instance is m = 2 -- *i.e.* what is usually called **parabolic** or **quadratic** material -- wherein

$$\frac{d^2}{dz^2}\,\rho \approx -\frac{2\Delta}{a}\left(\frac{\rho}{a}\right)^{2-1} = -\frac{2\Delta}{a^2}\,\rho = -\kappa^2\,\rho \qquad \text{[II-21]}$$

so that

$$\rho(z) = \rho_{\text{in}}\,\cos(\kappa z) + \frac{\rho'_{\text{in}}}{\kappa}\,\sin(\kappa z) \qquad \text{[II-22a]}$$

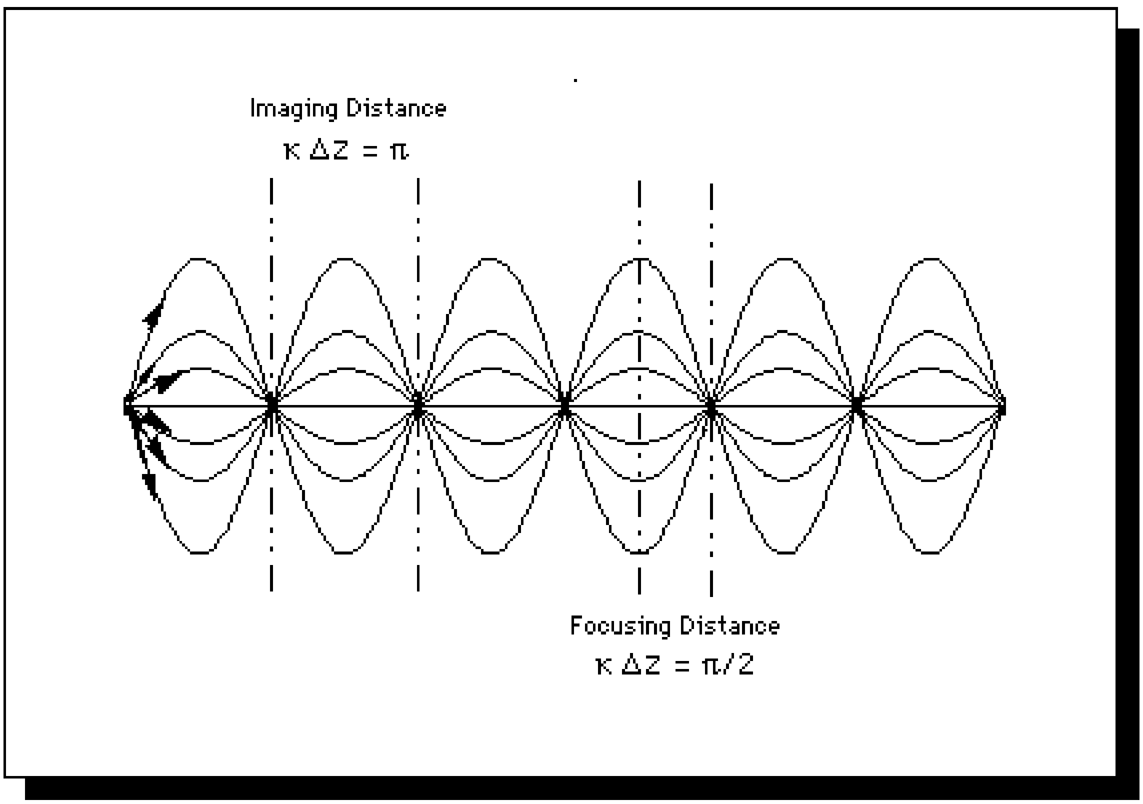
$$\rho'(z) = -\rho_{in}\,\kappa\,\sin(\kappa z) + \rho'_{in}\,\cos(\kappa z) \qquad \text{[II-22b]}$$

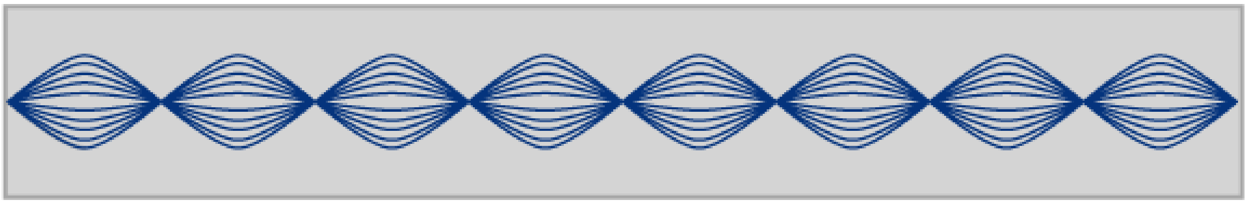
In terms of a ray transform matrix

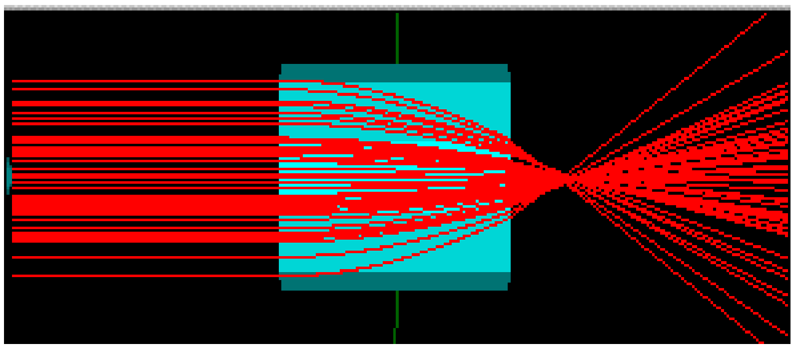
$$\begin{bmatrix} \rho_{out} \\ \rho'_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \rho_{in} \\ \rho'_{in} \end{bmatrix} \approx \begin{bmatrix} \cos(\kappa z) & \kappa^{-1}\sin(\kappa z) \\ -\kappa\sin(\kappa z) & \cos(\kappa z) \end{bmatrix} \begin{bmatrix} \rho_{in} \\ \rho'_{in} \end{bmatrix} \qquad \text{[II-23]}$$

where $\kappa = \sqrt{2\Delta / a^2}$.

Ray trajectories confined in a GRIN rod.







ALTERNATIVE (HAMILTONIAN) DERIVATION OF EIKONAL EQUATION:

FERMAT'S PRINCIPLE [10]

Like most laws of physics, the equations of geometric optics can be derived from a **variation principle**. In this context the variation principle is called the Fermat principle which states that a ray always chooses a trajectory that minimizes[11] the optical path length -- *viz*.

$$\int_{P_1}^{P_2} n\left(x,y,z\right)\, ds = \text{minimum} \qquad \text{[II-24]}$$

10 See, for example, Dietrich Marcuse, *Light Transmission Optics*, Van Nostrand Reinhold (1972).

11 More precisely, the path must be a local *extremum* and in rare cases may, in fact, be a maximum. See R. Y. Luneberg, *Mathematical Theory of Optics*, University of California Press, Berkeley and Los Angeles (1964).

where the line element, ds, is measured along a ray and the two end-points P_1 and P_2 are fixed in space.[12] Analysis of the variation problem is facilitated by choosing the projected coordinate z as the new variable of integration. Accordingly,

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{1 + x'^2 + y'^2}\; dz \quad , \qquad \text{[II-25]}$$

where $x' = \dfrac{dx}{dz}$ and $y' = \dfrac{dy}{dz}$, Fermat's variation principle is transformed into the more familiar **Lagrangian form** -- *viz.*

$$\int_{P_1}^{P_2} L(x, y, x', y')\, dz = \text{minimum} \qquad \text{[II-26]}$$

where

$$L(x, y, x', y') = n(x, y, z)\,\sqrt{1 + x'^2 + y'^2}\ . \qquad \text{[II-27]}$$

The minimization procedure is then well-known in the ***variational calculus*** and leads to the famous **Euler-Lagrangian equations** -- *i.e.*

$$\frac{d}{dz}\frac{\partial L}{\partial x'} - \frac{\partial L}{\partial x} = 0 \qquad \text{[II-28a]}$$

$$\frac{d}{dz}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y} = 0 \qquad \text{[II-28b]}$$

When applied to the **Fermat Lagrangian**, as defined in Equation [II-27], these equations yield

---

12 From Equation [II-7] we see that

$$S(P_2) - S(P_1) = \int_{P_1}^{P_2} n\,(x,y,z)\; ds\ .$$

$$\frac{d}{dz}\frac{n x'}{\sqrt{1+x'^2+y'^2}} = \sqrt{1+x'^2+y'^2}\,\frac{\partial n}{\partial x} \qquad \text{[II-29a]}$$

$$\frac{d}{dz}\frac{n y'}{\sqrt{1+x'^2+y'^2}} = \sqrt{1+x'^2+y'^2}\,\frac{\partial n}{\partial y}\ . \qquad \text{[II-29b]}$$

Using Equation [II-25] we see that the **Euler-Lagrangian** equations may be expressed in the vector form as

$$\left\{ \frac{d}{ds}\left[n\frac{dx}{ds}\right], \frac{d}{ds}\left[n\frac{dy}{ds}\right]\right\} = \left\{ \frac{\partial n}{\partial x}, \frac{\partial n}{\partial y}\right\} \qquad \text{[II-30]}$$

which is precisely the content of Equation [II-9] -- QED.

HAMILTONIAN FORMULATION OF RAY OPTICS

The analogy between ray optics and particle mechanics is most striking when the equations of ray optics are expressed in Hamiltonian form.[13] To that end, we define the **generalized momentum** which is canonically conjugate to x and y by the vector equation

$$\left\{ p_x, p_y \right\} = \left\{ \frac{\partial L}{\partial x}, \frac{\partial L}{\partial y} \right\}\ . \qquad \text{[II-31]}$$

The Hamiltonian is then define in terms of the generalized momentum by the relation

$$H\left(x,y,p_x,p_y\right) \equiv p_x x' + p_y y' - L\left(x,y,x',y'\right)\ . \qquad \text{[II-32]}$$

With the assumed functional dependence of the Hamiltonian, we form the derivatives

13 The formal theory of optical systems was developed by Sir W. R. Hamilton in 1828-37.

$$\frac{\partial H}{\partial p_x} = x' + p_x \frac{\partial x'}{\partial p_x} + p_y \frac{\partial y'}{\partial p_x} - \frac{\partial L}{\partial x'}\frac{\partial x'}{\partial p_x} - \frac{\partial L}{\partial y'}\frac{\partial y'}{\partial p_x} \qquad \text{[II-33a]}$$

$$\frac{\partial H}{\partial p_y} = p_x \frac{\partial x'}{\partial p_y} + y' + p_y \frac{\partial y'}{\partial p_y} - \frac{\partial L}{\partial x'}\frac{\partial x'}{\partial p_y} - \frac{\partial L}{\partial y'}\frac{\partial y'}{\partial p_y} \ . \qquad \text{[II-33b]}$$

Given the definitional relationships embodied in Equation [II-31] we see that these expression reduce to one set of Hamilton's equation -- *viz*.

$$\left\{ \frac{dx}{dz}, \frac{dy}{dz} \right\} = \left\{ \frac{\partial H}{\partial p_x}, \frac{\partial H}{\partial p_y} \right\} . \qquad \text{[II-34]}$$

The other set of Hamilton's equation -- *viz*.

$$\left\{ \frac{d\,p_x}{dz}, \frac{d\,p_y}{dz} \right\} = \left\{ -\frac{\partial H}{\partial x}, -\frac{\partial H}{\partial y} \right\} . \qquad \text{[II-35]}$$

follow directly from the Euler-Lagrangian equations -- *i.e.* Equations [II-28a] and [II-28b] -- and the definitions embodied in Equation [II-31]. Using the Fermat Lagrangian we see that

$$\left\{ \ p_x , p_y \right\} = \left\{ \ \frac{\partial L}{\partial x}, \frac{\partial L}{\partial y} \ \right\} = \left\{ \ \frac{n\,x'}{\sqrt{1+x'^2+y'^2}}, \frac{n\,y'}{\sqrt{1+x'^2+y'^2}} \ \right\} \qquad \text{[II-36]}$$

and consequently that we may solve for $\left\{ \ x', y' \ \right\}$ in terms of $\left\{ \ p_x , p_y \ \right\}$ as

$$\left\{ \ x', y' \ \right\} = \left\{ \ \frac{p_x}{\sqrt{n^2 - p_x^2 - p_y^2}}, \frac{p_y}{\sqrt{n^2 - p_x^2 - p_y^2}} \ \right\} \qquad \text{[II-37]}$$

Substituting into Equation [II-32], we find an expression for the **Fermat** or **ray optics Hamiltonian** -- *viz.*

$$H = -\sqrt{n^2 - p_x^2 - p_y^2} \quad . \qquad \text{[II-38]}$$

which resembles the mechanical Hamilton of a relativistic particle -- *i.e.*,

$$c\sqrt{m_0^2 c^2 + p_x^2 + p_y^2 + p_z^2} \quad .$$

But the analogy is even stronger in the paraxial approximation where the Hamiltonian is approximated by an expression which is identical in form with the Hamiltonian of a non-relativistic particle -- *viz.*

$$H = -n\sqrt{1 - \frac{p_x^2 + p_y^2}{n^2}} \approx \frac{p_x^2 + p_y^2}{2\langle n \rangle} - n \qquad \text{[II-39]}$$

when p_x and $p_y << \langle n \rangle$. [14]

[14] Applying the quantization rules of quantum mechanics to these Hamiltonians, we can go full circle and recover wave optics from ray optics. Equation [II-38] leads directly to the equivalent of the relativistic Klein-Gordon equation while the equivalent of the nonrelativistic Schrödinger equation follows directly from Equation [II-39].

III. THE PARAXIAL WAVE EQUATION -- PROPAGATION OF GAUSSIAN BEAMS IN UNIFORM MEDIA

DERIVATION OF PARAXIAL WAVE EQUATION:

In point-to-point communication, we may think of the electromagnetic field as propagating in a kind of "searchlight" mode -- *i.e.* a beam of finite width that propagates in some particular direction. In analyzing this mode of wave propagation, we make use of an important solution to the so call paraxial approximation of the electromagnetic wave equation (or, more precisely, the paraxial approximation of the Helmholz equation).

To that end, we first derive the paraxial approximation and then examine the free-space ***Gaussian Beam*** solution(s). We start with the homogeneous Helmholz equation for the vector potential in the form -- see Equation [I-13a]

$$ {}^{2}\,\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)+\omega^{2}\,\mu_{0}\,\varepsilon(\omega)\,\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)= \;{}^{2}\,\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)+k^{2}\,\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)=0 \qquad \text{[III-1]}$$

We are looking for a wave propagating in, say, the z-direction, so we write a particular component of the potential in the form

$$ \mathrm{A}_{\alpha}(\vec{\mathbf{r}},\omega)=\Psi(\vec{\mathbf{r}},\omega)\exp(-ikz) \qquad \text{[III-2]}$$

The function $\Psi(\vec{\mathbf{r}},\omega)$ represents a spatial modulation or "masking" of a plane wave propagating in the z-direction. The z-direction is obviously special and it is useful to appropriately parse the differential operators. For the **grad** operator we may write

$$ grad\{\qquad\}=\vec{\ }\{\qquad\}=\vec{\ }_{t}\{\qquad\}+\hat{\mathbf{z}}\frac{\partial}{\partial z}\{\qquad\} \qquad \text{[III-3]}$$

where, for example,

$$\vec{\nabla}_t\{\qquad\} = \hat{\mathbf{x}}\frac{\partial}{\partial x}\{\qquad\} + \hat{\mathbf{y}}\frac{\partial}{\partial y}\{\qquad\}\ . \qquad [\text{ III-4 }]$$

so that

$$\vec{\nabla} A_\alpha(\vec{\mathbf{r}},\omega) = \left[\vec{\nabla}_t\,\Psi(\vec{\mathbf{r}},\omega) + \hat{\mathbf{z}}\frac{\partial}{\partial z}\,\Psi(\vec{\mathbf{r}},\omega) - ik\,\hat{\mathbf{z}}\,\Psi(\vec{\mathbf{r}},\omega)\right]\exp(-ik\,z) \qquad [\text{ III-5 }]$$

For the **Laplacian** operator we may write

$$\nabla^2 A_\alpha(\vec{\mathbf{r}},\omega) = \nabla_t^2\,\Psi(\vec{\mathbf{r}},\omega)\,\exp(-ik\,z) + \frac{\partial}{\partial z}\left\{\left[\frac{\partial}{\partial z}\,\Psi(\vec{\mathbf{r}},\omega) - ik\,\Psi(\vec{\mathbf{r}},\omega)\right]\exp(-ikz)\right\} \qquad [\text{ III-6 }]$$

where, for example,

$$\nabla_t^2\{\qquad\} = \frac{\partial^2}{\partial x^2}\{\qquad\} + \frac{\partial^2}{\partial y^2}\{\qquad\} \qquad [\text{ III-7 }]$$

Therefore,

$$\nabla^2 A_\alpha(\vec{\mathbf{r}},\omega) = \left[\nabla_t^2\,\Psi(\vec{\mathbf{r}},\omega) + \frac{\partial^2}{\partial z^2}\,\Psi(\vec{\mathbf{r}},\omega) - 2ik\,\frac{\partial}{\partial z}\,\Psi(\vec{\mathbf{r}},\omega) - k^2\,\Psi(\vec{\mathbf{r}},\omega)\right]\exp(-ikz) \qquad [\text{ III-8 }]$$

and the parsed Helmholz equation (**without approximation**) becomes

$$\nabla_t^2\,\Psi(\vec{\mathbf{r}},\omega) - 2ik\,\frac{\partial}{\partial z}\,\Psi(\vec{\mathbf{r}},\omega) + \frac{\partial^2}{\partial z^2}\,\Psi(\vec{\mathbf{r}},\omega) = 0 \qquad [\text{ III-9 }]$$

The **paraxial approximation** is precisely defined by the condition

$$2ik\,\frac{\partial}{\partial z}\,\Psi(\vec{\mathbf{r}},\omega) >> \frac{\partial^2}{\partial z^2}\,\Psi(\vec{\mathbf{r}},\omega) \qquad [\text{ III-10 }]$$

which means that the longitudinal variation in the modulation function, $\Psi(\vec{\mathbf{r}},\omega)$, changes very little in the wavelength associated with beam -- *i.e.* $2\pi/k$. In this approximation, we neglect the third term and obtain the equation

$$\boxed{\nabla_t^2\,\Psi(\vec{\mathbf{r}},\omega) - 2ik\,\frac{\partial}{\partial z}\,\Psi(\vec{\mathbf{r}},\omega) = 0} \qquad \text{[III-11]}$$

which is called the **paraxial approximation** of the wave equation.[15]

SOLUTIONS OF THE PARAXIAL WAVE EQUATION

The Gaussian beam

To inform or motivate our next step, we consider the paraxial approximation of a known solution of the Helmholz equation -- *i.e.* a spherical wave

$$\frac{\exp(-ikr)}{r} = \frac{\exp\left(-ik\sqrt{x^2+y^2+z^2}\right)}{\sqrt{x^2+y^2+z^2}} = \frac{\exp\left(-ikz\sqrt{1+\dfrac{x^2+y^2}{z^2}}\right)}{z\sqrt{1+\dfrac{x^2+y^2}{z^2}}} \approx \frac{\exp(-ikz)\exp\left[\dfrac{-ik\left(x^2+y^2\right)}{2z}\right]}{z} \qquad \text{[III-12]}$$

Reflecting on the "quadratic" form of this approximate expression, it is reasonable to look for an axially symmetric solution of the paraxial wave equation in the following form -- *i.e.* a **Gaussian beam**:

$$\Psi_G(\rho,z,\omega) = A_G\,\exp[-i\,P(z)]\,\exp\left[-\frac{ik\rho^2}{2q(z)}\right] \qquad \text{[III-13]}$$

15 Obvious the paraxial equation has the same mathematical form as the Schrödinger equation and, thus, all that is know about solutions of that equation may be directly applied to understand issues in light propagation (or visa versa).

where $\rho^2 = x^2 + y^2$.

We test our conjecture by substituting the Gaussian beam function -- *i.e.* Equation [III-13] -- into the paraxial wave equation -- *i.e.* Equation [III-11] -- to wit

$$\exp\left[-i\,P(z)\right]\ \nabla_t^2 \left\{ \exp\left[-\frac{ik\,\rho^2}{2q(z)}\right]\right\} - 2ik\,\frac{\partial}{\partial z}\left\{\exp\left[-iP(z)\right]\exp\left[-\frac{ik\,\rho^2}{2q(z)}\right]\right\} = 0 \ . \qquad [\text{ III-14 }]$$

Executing the indicated operations, we obtain

$$\exp\left[-i\,P(z)\right]\exp\left[-\frac{ik\,\rho^2}{2q(z)}\right]\left\{-\frac{2\,ik}{q(z)} - \frac{k^2\rho^2}{\left[q(z)\right]^2} - 2\,ik\left[-i\frac{\partial}{\partial z}P(z) + \frac{ik\,\rho^2}{2\left[q(z)\right]^2}\frac{\partial}{\partial z}q(z)\right]\right\} = 0 \qquad [\text{ III-15a }]$$

or simplifying

$$2\,k\frac{\partial}{\partial z}P(z) + \frac{2ik}{q(z)} + \frac{k^2\rho^2}{\left[q(z)\right]^2}\left\{1 - \frac{\partial}{\partial z}q(z)\right\} = 0 \ . \qquad [\text{ III-15b }]$$

Hence, for an **arbitrary** ρ this equation is separable into two parts -- *viz.*

$$\frac{k^2\rho^2}{\left[q(z)\right]^2}\left\{1 - \frac{\partial}{\partial z}q(z)\right\} = 0 \longrightarrow \frac{\partial}{\partial z}q(z) = 1 \qquad [\text{ III-15c }]$$

and

$$2\,k\frac{\partial}{\partial z}P(z) + \frac{2ik}{q(z)} = 0 \longrightarrow \frac{\partial}{\partial z}P(z) = \frac{-i}{q(z)} \qquad [\text{ III-15d }]$$

which are satisfied by the simple solutions

$$q(z) = z + q_0 \qquad [\text{ III-16a }]$$

and

$$\frac{\partial}{\partial z} P(z) = \frac{-i}{z+q_0} = -i \frac{\partial}{\partial z} \ln\left[z+q_0\right] \longrightarrow P(z) = -i \ln\left[z+q_0\right] . \qquad [\text{ III-16b }]$$

On comparison with the paraxial approximation of a spherical wave -- *i.e* Equation [III-12] -- we may write $q(z)$ in terms of a radius of curvature $R(z)$ and a width $w(z)$ -- *viz.*

$$\frac{1}{q(z)} = \frac{1}{z+q_0} = \frac{1}{R(z)} + \frac{-i\,2}{k\,w^2(z)} . \qquad [\text{ III-17 }]$$

To standardize the constants of integration we assume a **plane wavefront** at an arbitrary reference point $z = 0$ -- *i.e.* we take $R(0) \equiv \infty$. Thus,

$$\frac{1}{R(0)} = 0 \qquad [\text{ III-18a }]$$

and

$$\frac{-i\,2}{k\,w^2(0)} \equiv \frac{1}{q_0} \qquad q_0 = \frac{i\,k\,w^2(0)}{2} = \frac{i\,\pi\,w^2(0)}{\lambda} = i\,L_F \qquad [\text{ III-18b }]$$

where $L_F = k\,w^2(0)/2 = \pi\,w^2(0)/\lambda$ is the critical Gaussian beam scaling parameter which is called variously the **Fresnel length**, the **diffraction length**, or the **confocal parameter**. In terms of this parameter, Equation [III-17] may be written

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{-i\,2}{k\,w^2(z)} = \frac{1}{z+i\,L_F} = \frac{z-i\,L_F}{z^2+L_F^2} \qquad [\text{ III-19 }]$$

Equating real and imaginary parts, we obtain

$$\frac{1}{R(z)} = \frac{z}{z^2+L_F^2} \qquad \text{and} \qquad \frac{-i\,2}{k\,w^2(z)} = \frac{-iL_F}{z^2+L_F^2}$$

or, finally, in standardized form

$$\boxed{\begin{aligned} R(z) &= z\left[1 + L_{\mathrm{F}}^2/z^2\right] \\ w^2(z) &= w^2(0)\left[1 + z^2/L_{\mathrm{F}}^2\right] \\ &\text{where} \quad L_{\mathrm{F}} = \pi\, w^2(0)/\lambda \end{aligned}} \qquad \text{[III-20]}$$

Now since Equation [III-16b] may be written

$$P(z) = -i\,\ln\left[z + q_0\right] = -i\,\ln\left[z + i\,L_{\mathrm{F}}\right] = -i\left\{\ln\left[z^2 + L_{\mathrm{F}}^2\right] + i\,\tan^{-1}\left[L_{\mathrm{F}}/z\right]\right\} l \qquad \text{[III-16b']}$$

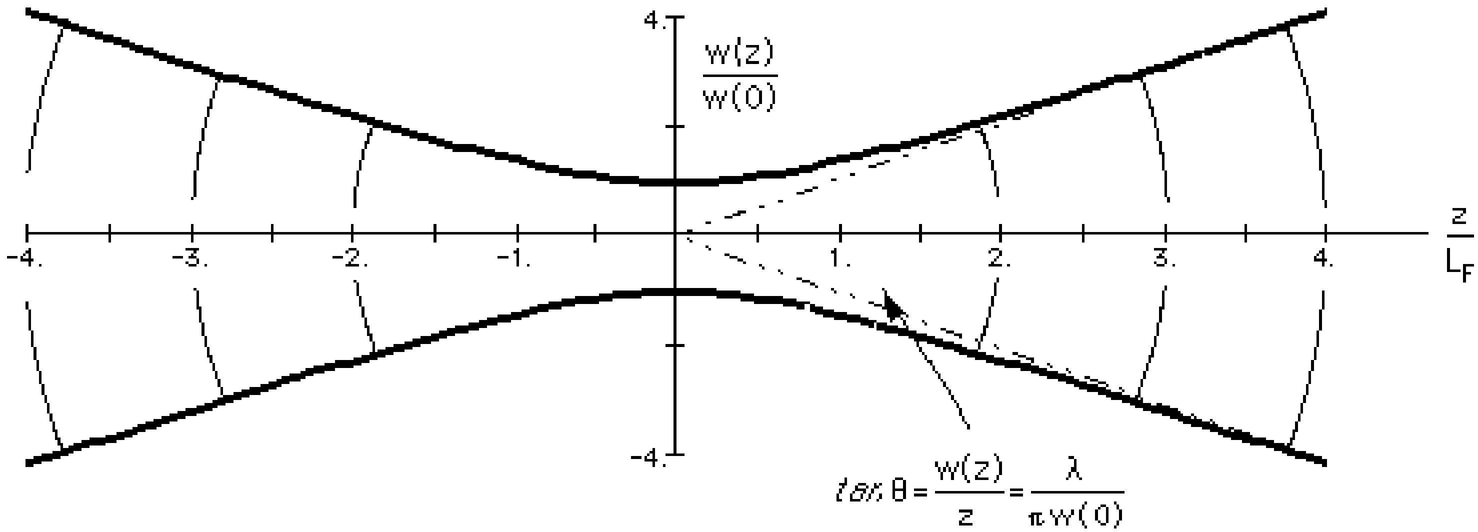
we may write

$$\exp\left[-i\,P(z)\right] = \frac{\exp\left[-i\,\tan^{-1}\left[L_{\mathrm{F}}/z\right]\right]}{\sqrt{z^2 + L_{\mathrm{F}}^2}} = \frac{\exp\left[-i\,\tan^{-1}\left[L_{\mathrm{F}}/z\right]\right]}{z\sqrt{1 + L_{\mathrm{F}}^2/z^2}}$$

to obtain the usual, **officially approved** form of the Gaussian Beam

$$\begin{aligned} \Psi_{\mathrm{G}}(\rho, z, \omega) &= A_{\mathrm{G}}\,\exp\left[-i\,P(z)\right]\exp\left[-\frac{ik\rho^2}{2\,q(z)}\right] \\ &= A_{\mathrm{G}}\,\frac{w(0)}{w(z)}\,\exp\left[-i\,\tan^{-1}\left[L_{\mathrm{F}}/z\right]\right]\exp\left[-\frac{ik\rho^2}{2\,R(z)}\right]\exp\left[-\frac{\rho^2}{w^2(z)}\right] \end{aligned} \qquad \text{[III-21]}$$

The following kind of picture is sometimes found to be helpful in understanding the propagation of a Gaussian Beam (the bold curve depicts the spatial variation of the beam width and the light curve the beam curvature at particular points in space):



Higher order Hermite-Gaussian beams

In order to study the propagation of higher order beams, we substitute the following trial solution:

$$\begin{aligned}\Psi_{\text{H-G}}(\rho, z, \omega) &= F(x, y, z)\,\Psi_{\text{G}}(\rho, z, \omega)\\ &= f\left(\frac{x}{w}\right) g\left(\frac{y}{w}\right) \exp\left[-i\,\Phi(z)\right]\,\Psi_{\text{G}}(\rho, z, \omega)\end{aligned} \qquad \text{[III-22]}$$

into the paraxial wave equation -- *viz.* Equation [III-11] -- and obtain

$$\begin{aligned} &\mathrm{F}(x,y,z)\,\nabla_t^2\,\Psi_G(\rho,z,\omega)+2\left[\vec{\nabla}_t \mathrm{F}(x,y,z)\cdot\vec{\nabla}_t\Psi_G(\rho,z,\omega)\right]+\Psi_G(\rho,z,\omega)\,\nabla_t^2\,\mathrm{F}(x,y,z) \\ &\qquad -2ik\,\Psi_G(\rho,z,\omega)\frac{\partial}{\partial z}\mathrm{F}(x,y,z)-2ik\,\mathrm{F}(x,y,z)\frac{\partial}{\partial z}\Psi_G(\rho,z,\omega)=0 \end{aligned} \qquad \text{[III-23]}$$

Since the sum of the first and fifth terms already satisfies the paraxial wave equation, Equation [III-23] reduces to

$$\frac{\mathrm{f}''}{\mathrm{f}}+2ik\left[\frac{dw}{dz}-\frac{w}{q}\right]x\,\frac{\mathrm{f}'}{\mathrm{f}}+\frac{\mathrm{g}''}{\mathrm{g}}+2ik\left[\frac{dw}{dz}-\frac{w}{q}\right]y\,\frac{\mathrm{g}'}{\mathrm{g}}-2\,k\,w^2\,\frac{d\Phi}{dz}=0 \qquad \text{[III-24]}$$

From Equations [III-19] and [III-20] we see that

$$\frac{dw}{dz}-\frac{w}{q}=\frac{w}{R}-\left(\frac{w}{R}+\frac{-i\,2}{k\,w}\right)=\frac{i\,2}{k\,w}$$

so that the reduced equation -- *i.e.* Equation [III-24] -- becomes

$$\frac{\mathrm{f}''}{\mathrm{f}}-4\,\xi\,\frac{\mathrm{f}'}{\mathrm{f}}+\frac{\mathrm{g}''}{\mathrm{g}}-4\,\varsigma\,\frac{\mathrm{g}'}{\mathrm{g}}-2\,k\,w^2\,\frac{d\Phi}{dz}=0 \qquad \text{[III-25]}$$

where $\xi = x/w$ and $\varsigma = y/w$.

A **Hermite polynomial** of order n[16] has the following differential equation:

$$\frac{d^2}{d\tau^2}\mathrm{H}_n(\tau)-2\tau\,\frac{d}{d\tau}\mathrm{H}_n(\tau)+2n\,\mathrm{H}_n(\tau)=0. \qquad \text{[III-26]}$$

16 The Hermite polynomials have the generator $\mathrm{H}_n(\tau)=(-1)^n\,\exp\left(\tau^2\right)\frac{d^n}{d\tau^n}\exp\left(-\tau^2\right)$.

With the simple change in variables $\tau = \sqrt{2}\,\xi = \sqrt{2}\;x/w$ and $\sigma = \sqrt{2}\,\varsigma = \sqrt{2}\;y/w$ Equation [III-25] may be written

$$\frac{1}{f}\left[\frac{d^2 f}{d\tau^2} - 2\tau\,\frac{d f}{d\tau}\right] + \frac{1}{g}\left[\frac{d^2 g}{d\sigma^2} - 2\sigma\,\frac{d g}{d\sigma}\right] - 2\,kw^2\,\frac{d\,\Phi}{d\,z} = 0 \qquad \text{[III-27]}$$

Thus, it is apparent that we can write the functions $f\left(\frac{x}{w}\right)$ and $g\left(\frac{y}{w}\right)$ as Hermite polynomials -- *viz.*

$$f\left(\frac{x}{w}\right) = \mathrm{H}_n(\tau) = \mathrm{H}_n\left(\sqrt{2}\;\frac{x}{w}\right) \quad \text{and} \quad g\left(\frac{x}{w}\right) = \mathrm{H}_m(\sigma) = \mathrm{H}_m\left(\sqrt{2}\;\frac{y}{w}\right)$$

if we require that $2\,kw^2\,\frac{d\,\Phi}{dz} = -2(n+m)$. Hence

$$\frac{d\,\Phi}{d\,z} = -\frac{(n+m)}{kw^2} = -\frac{(n+m)\,L_{\mathrm{F}}}{2\left[L_{\mathrm{F}}^2 + z^2\right]}$$

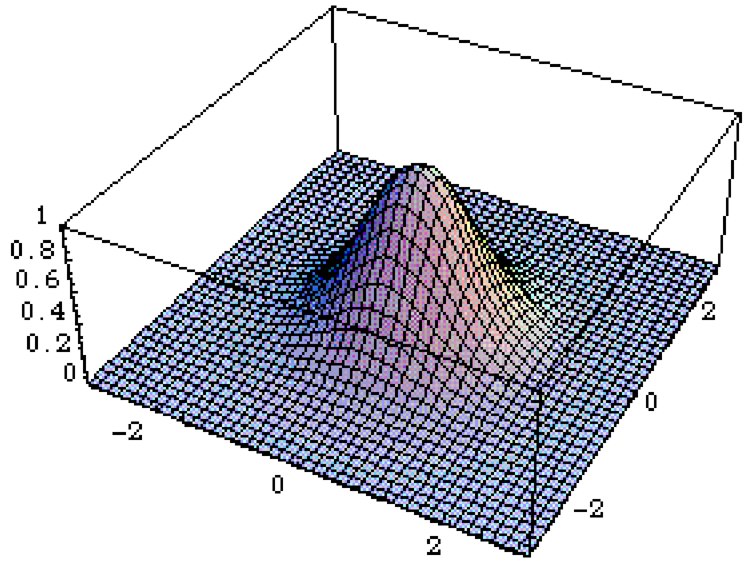
or

$$\Phi(z) = -(n+m)\;a\tan\left(z/L_{\mathrm{F}}\right).$$

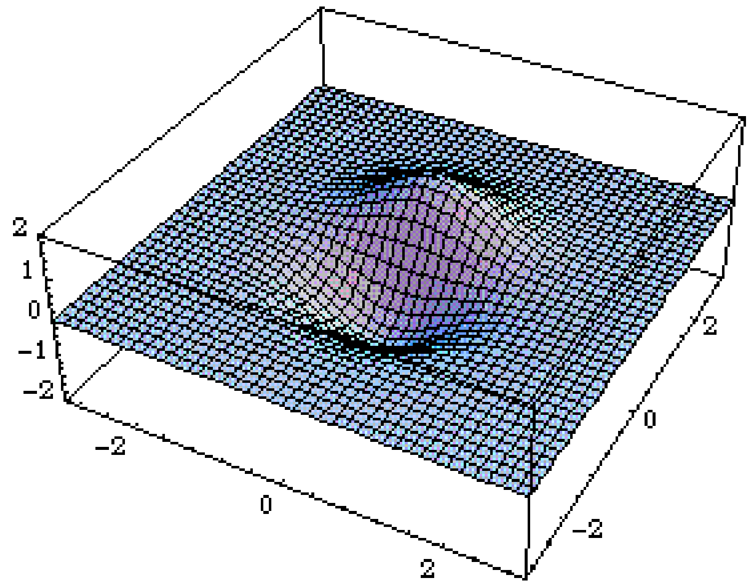
Finally we may write a general solution for the paraxial equation as

$$\begin{aligned}\Psi_{\text{H-G}}^{\text{nm}}(\rho, z, \omega) = A_{\text{H-G}}^{\text{nm}}\;\frac{w(0)}{w(z)}\;\mathrm{H}_n\left(\sqrt{2}\;\frac{x}{w}\right)\mathrm{H}_m\left(\sqrt{2}\;\frac{y}{w}\right)& \\ \times \exp\left[i\,[n+m+1]\,\tan^{-1}\left(z/L_{\mathrm{F}}\right)\right]\exp\left[-\frac{ik\,\rho^2}{2\,R(z)}\right]\exp\left[-\frac{\rho^2}{w^2(z)}\right]&\end{aligned} \qquad \text{[III-28]}$$

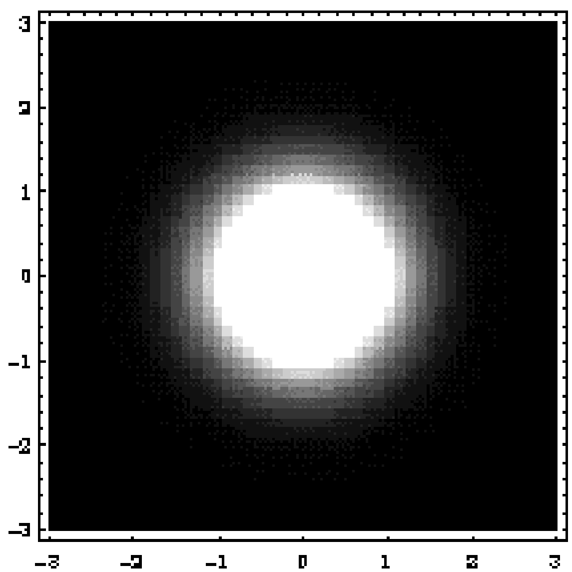
A GALLERY OF HERMITE-GAUSSIAN FIELD DISTRIBUTIONS

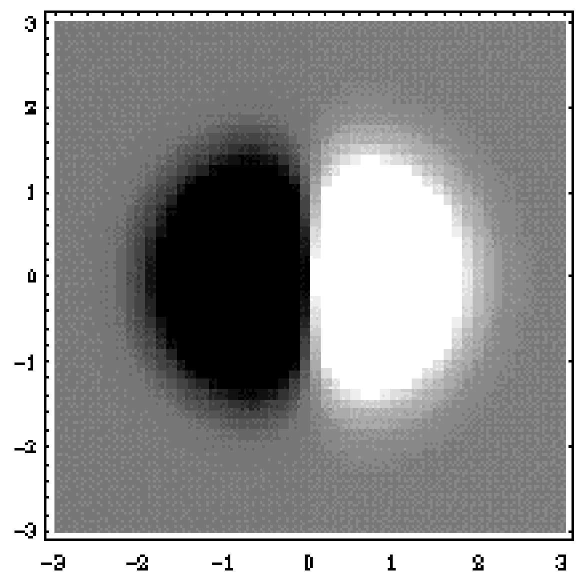


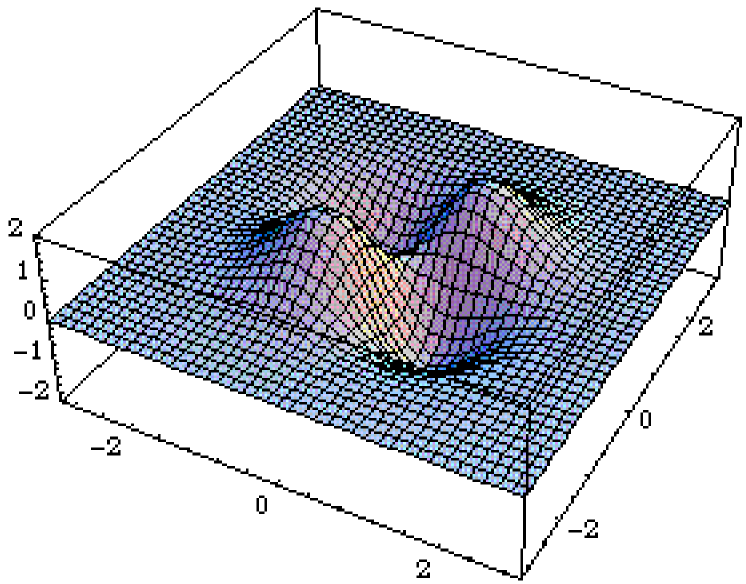
[0, 0] Hermite-Gaussian

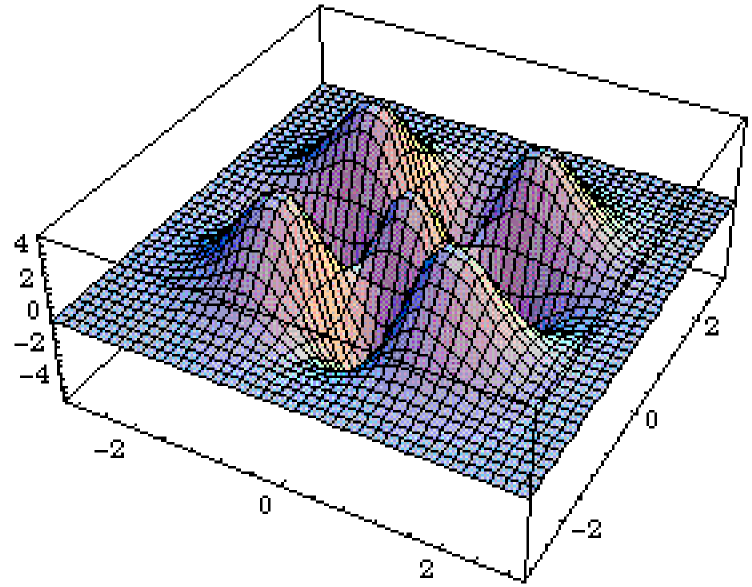


[0, 1] Hermite-Gaussian



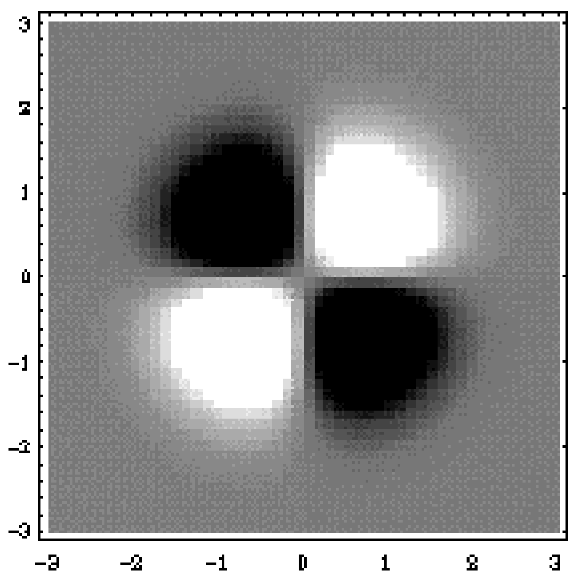


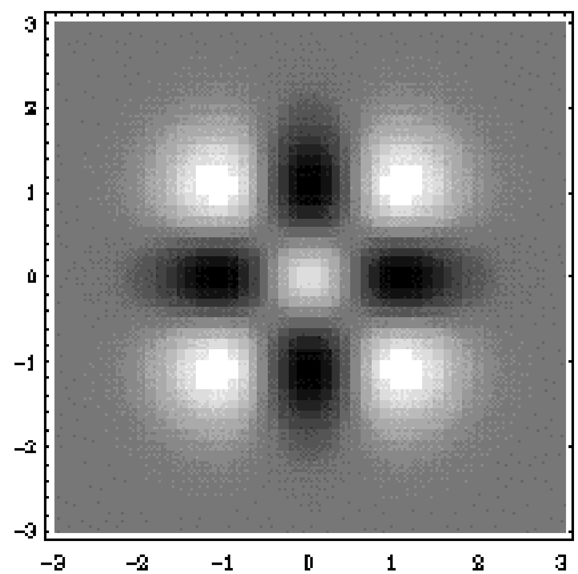




[1, 1] Hermite-Gaussian

[2, 2] Hermite-Gaussian





GAUSSIAN BEAM TRANSFORMATION MATRICES

What we have shown above is that a given Hermite-Gaussian beam is essentially completely specified or defined by the complex function $q(z)$. In propagating through an optical system, the beams are transformed by various optical components. **The amazing fact is that the transformation produced by a given component follows a simple ABCD transformation law** -- *viz.*

$$\boxed{q_2 = \frac{Aq_1 + B}{Cq_1 + D}} \qquad \text{[III-29]}$$

where A, B, C, D are the matrix elements found in our analysis of geometric optics!!

To **"prove"** this, we argue by example. For example, the transformation through a uniform dielectric region is given by

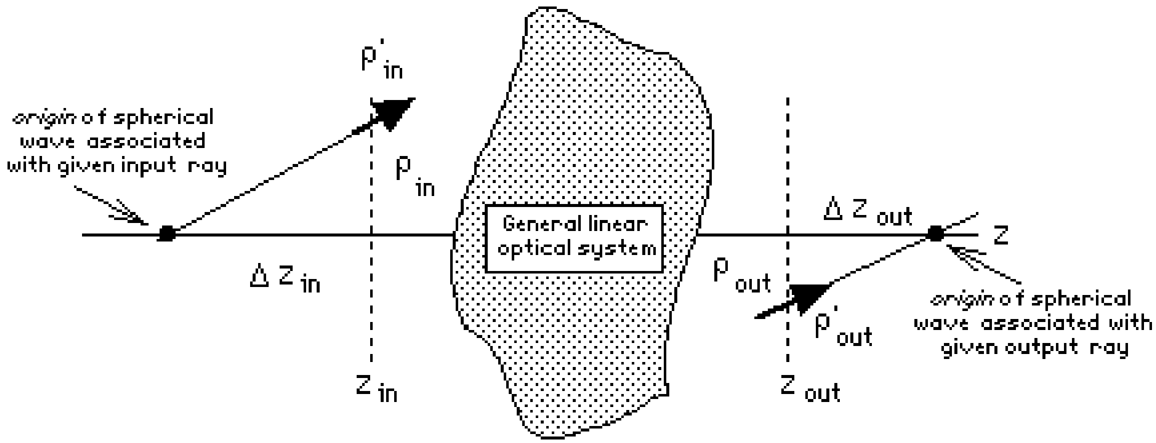
$$q_2 = q_1 + L$$

so that $\{A = 1;\; B = L;\; C = 0;\; D = 1\}$ and the transformation through a thin lens is given by

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f}$$

so that $\left\{A = 1;\; B = 0;\; C = -\frac{1}{f};\; D = 1\right\}$.

Further **"justification"** of this transformation law may be found in terms of the so called "ρ/ρ' argument"-- *viz*



From geometric optics and, in particular, Equation [II-11], we may write

$$\frac{\rho_{\mathrm{out}}}{\rho'_{\mathrm{out}}} = \left(A\,\frac{\rho_{\mathrm{in}}}{\rho'_{\mathrm{in}}} + B\right)\left(C\,\frac{\rho_{\mathrm{in}}}{\rho'_{\mathrm{in}}} + D\right)^{-1}$$

or

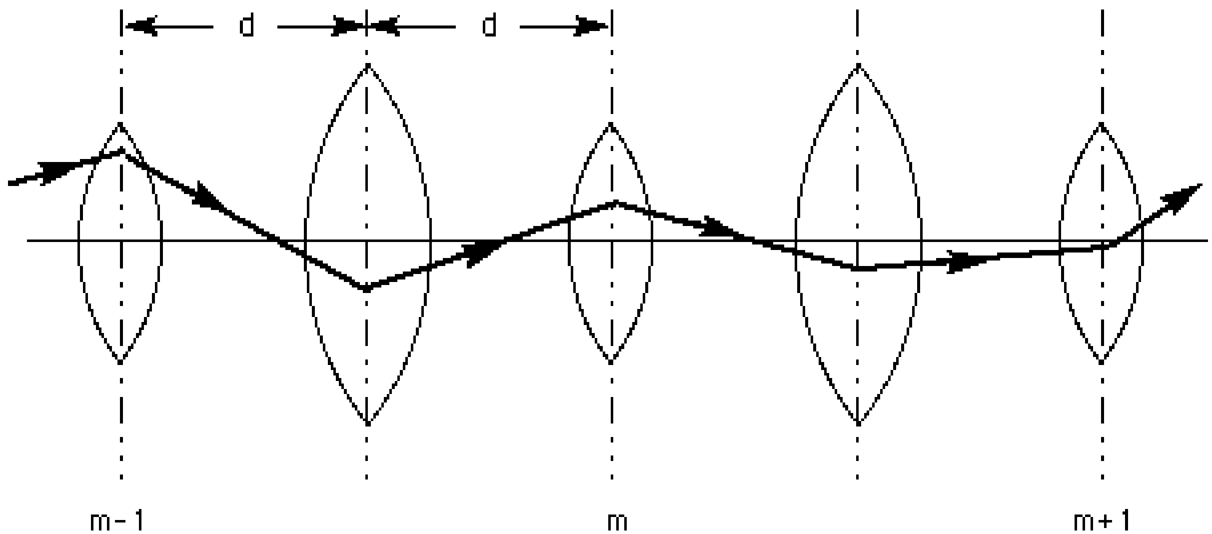
$$\Delta z_{\mathrm{out}} = \left(A\,\Delta z_{\mathrm{in}} + B\right)\left(C\,\Delta z_{\mathrm{in}} + D\right)^{-1}$$

which is identical to transformation equation [III-29] if we interpret $q(z)$ as the wave optics generalization of $\Delta z = \rho/\rho'$.

IV. OPTICAL RESONATORS:

STABILITY CRITERIA FOR RESONATORS AND PERIODIC OPTICAL STRUCTURES BY RAY OPTIC ANALYSIS

Consider a prototypical periodic guiding lens system or an equivalent resonator.



Using the appropriate **ABCD** matrix with the indicated reference planes [17], we may write

$$\rho_{m+1} = A\,\rho_m + B\,\rho'_m \qquad \text{[IV-1a]}$$

and

$$\rho'_{m+1} = C\,\rho_m + D\,\rho'_m \qquad \text{[IV-1b]}$$

---

[17] Where for reference , we see that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix} = \begin{bmatrix} 1-d/f_2-2\,d/f_1+d^2/f_1f_2 & 2d-d^2/f_2 \\ d/f_1f_2-1/f_1-1/f_2 & 1-d/f_2 \end{bmatrix}$$

From the first equation we write

$$\rho'_m = \frac{\rho_{m+1} - A\,\rho_m}{B} \quad \text{and} \quad \rho'_{m+1} = \frac{\rho_{m+2} - A\,\rho_{m+1}}{B} .$$

and substitute into Equation [IV-1b] to obtain

$$\rho_{m+2} - [A + D]\,\rho_{m+1} + [AD - BC]\,\rho_{m+1} = 0 \qquad \text{[IV-2a]}$$

The determinant of the coefficients $|A\,D - B\,C| = 1$ so that

$$\rho_{m+2} - [A + D]\,\rho_{m+1} + \rho_{m+1} = 0 \ . \qquad \text{[IV-2a]}$$

We see that

$$\frac{A+D}{2} = \beta = 1 - \frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{2\,f_1\,f_2} = -1 + 2\left(1 - \frac{d}{2\,f_1}\right)\left(1 - \frac{d}{2\,f_2}\right) \qquad \text{[IV-3]}$$

Thus, stable ray propagation may characterized by **bound solutions** of the form $\rho_m = \rho_0 \exp(i\,m\varphi)$ which are possible if and only if

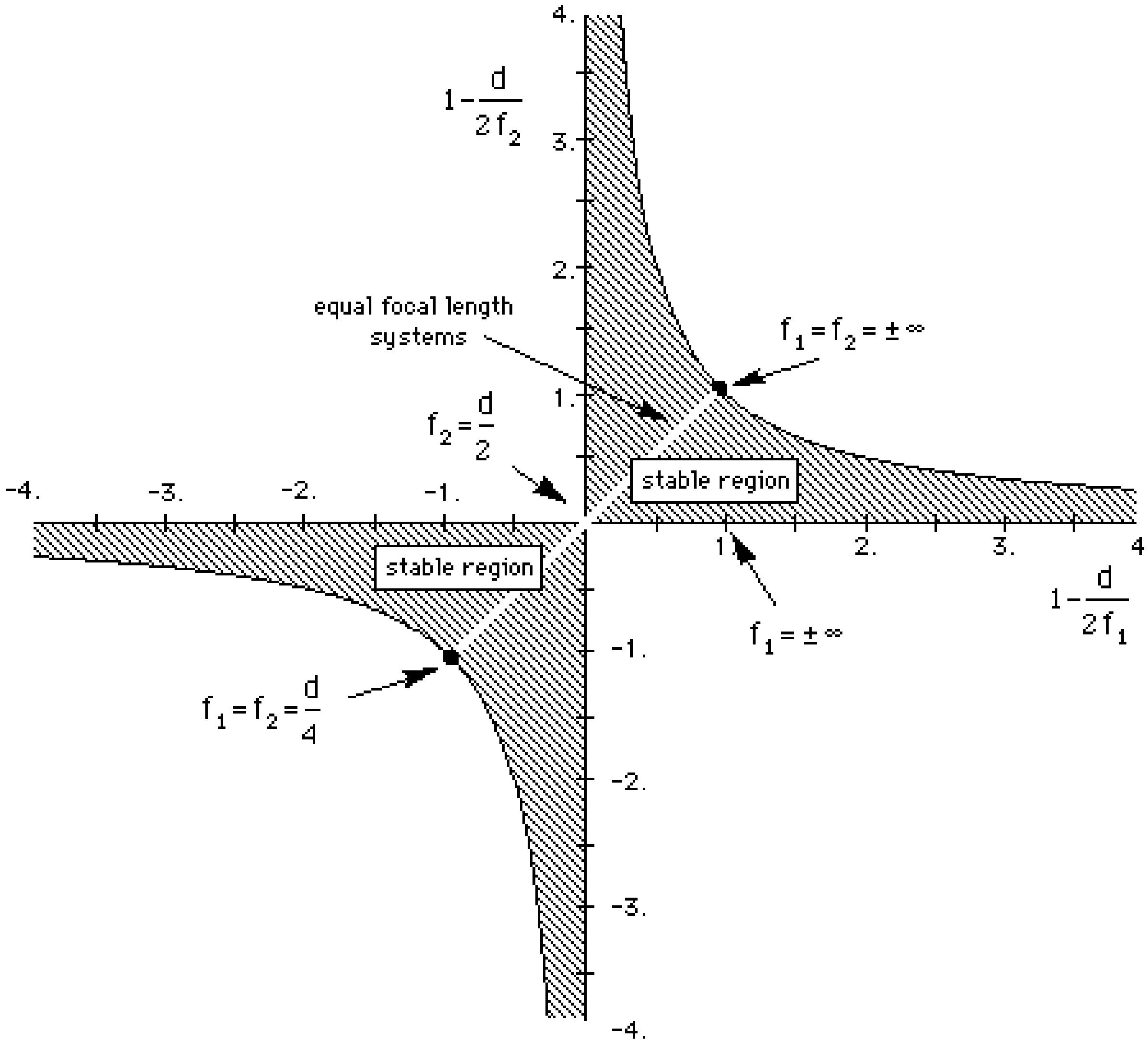
$$\exp(i\varphi) + \exp(-i\,\varphi) = 2\cos\varphi = A + D = 2\beta \quad \text{or} \quad \cos\varphi = \beta \qquad \text{[IV-4]}$$

Therefore propagation is stable -- *i.e.* the rays are confined -- when $|\,\beta\,| \le 1$ so that

$$0 \le \left(1 - \frac{d}{2\,f_1}\right)\left(1 - \frac{d}{2\,f_2}\right) \le 1 \ . \qquad \text{[IV-5]}$$

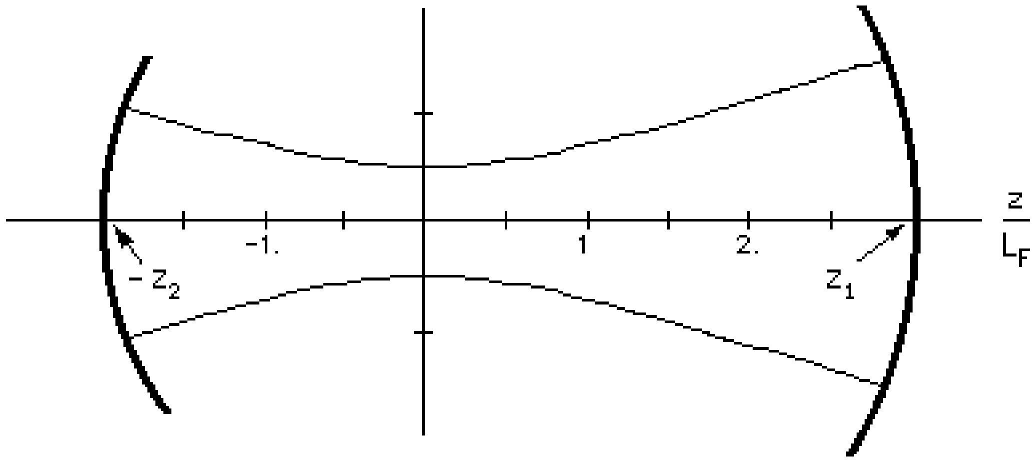
Ray stability of rays in a periodic system may be usefully characterized in terms of the variables $u_1 = 1 - \dfrac{d}{2\,f_1}$ and $u_2 = 1 - \dfrac{d}{2\,f_2}$ as follows:

Stability (Confinement) Diagram for Periodic Systems



STABILITY OF A SPHERICAL MIRROR RESONATORS -- USING SOLUTIONS OF THE PARAXIAL WAVE EQUATION:

Consider a Hermite-Gaussian mode confined in an **asymmetrical spherical cavity**:



In order to sustain a resonant mode in such a cavity, the beam's radius of curvature must match each mirror's radius of curvature at the mirror's surface and, thus, the following conditions must hold (see Equation [III-20]):

$$R_1 = z_1 + \frac{L_F^2}{z_1} \quad \text{and} \quad R_2 = -z_2 - \frac{L_F^2}{z_2} \qquad \text{[IV-5a]}$$

where $d = z_1 + z_2$. Hence, we see that

$$z_1 = \frac{R_1 \pm \sqrt{R_1^2 - 4L_F^2}}{2} \quad \text{and} \quad z_2 = \frac{-R_2 \pm \sqrt{R_2^2 - 4L_F^2}}{2} . \qquad \text{[IV-5b]}$$

with a lot of algebra we can show that

$$L_F^2 = \left[\frac{\pi w^2(0)}{\lambda}\right]^2 = \frac{-d u_1 u_2 R_1 R_2 \left[d + u_1 R_1 - u_2 R_2\right]}{\left[u_1 R_1 - u_2 R_2\right]^2} \qquad \text{[IV-6]}$$

where now $u_1 = 1 - d/R_1$ and $u_2 = 1 + d/R_2$.

For a **symmetric resonator** $R_2 = -R_1$ and $u_1 = u_2$

$$L_F^2 = \left[\frac{\pi w^2(0)}{\lambda}\right]^2 = \frac{d\left[2R - d\right]}{4} \qquad \text{[IV-7a]}$$

and

$$w(z_1) = w(-z_2) = \left[\frac{d\lambda}{2\pi}\right]^{1/2} \left[\frac{2R^2}{d\left(R - d/2\right)}\right]^{1/4} \qquad \text{[IV-7b]}$$

For an **asymmetric resonator**, it can be shown with a bit more algebra that

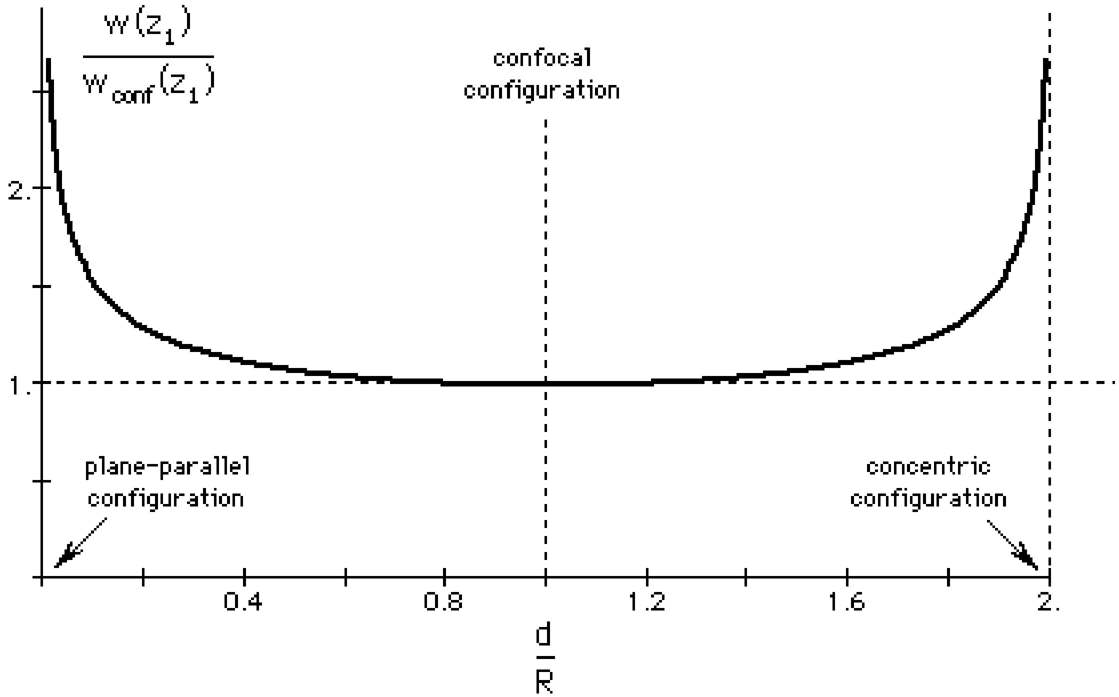
$$w(z_1) = \left[\frac{d\lambda}{\pi}\right]^{1/2} \left[\frac{u_2}{u_1\left(1 - u_1 u_2\right)}\right]^{1/4} \qquad \text{[IV-8a]}$$

$$w(-z_2) = \left[\frac{d\lambda}{\pi}\right]^{1/2} \left[\frac{u_1}{u_2\left(1 - u_1 u_2\right)}\right]^{1/4} \qquad \text{[IV-8b]}$$

As a measure of the effect of resonator length and mirror radius on **diffraction loss** consider the ratio:

$$\frac{w(z_1)}{w_{\mathrm{conf}}(z_1)} = \frac{w(-z_2)}{w_{\mathrm{conf}}(-z_2)} = \left[\frac{d}{R}\left(1 - \frac{d}{2R}\right)\right]^{-1/4} \qquad \text{[IV-9]}$$

where $w_{\text{conf}}\left(z_1\right) = \sqrt{2}\; w(0)$ is beam width at the mirror for the confocal configuration -- *i.e.*, when both mirrors have their focal points at the mid-point of the cavity



Resonance Frequencies of the Optical Resonator:

From Equation [III-28], we see the the "round-trip" cold resonance condition for a Hermite-Gaussian mode is given by

$$k\,d - (n + m + 1)\left[\tan^{-1}\left(z_1/L_F\right) + \tan^{-1}\left(z_2/L_F\right)\right] = N\pi \qquad \text{[IV-10a]}$$

where N is an integer. In terms of frequency, the resonce condition is

$$\omega = \frac{c}{d}\left\{ N\pi + (n + m + 1)\left[\tan^{-1}\left(z_1 / L_F\right) + \tan^{-1}\left(z_2 / L_F\right)\right]\right\} \qquad \text{[IV-10b]}$$

After much algebra, it can be shown that:

$$\omega = \frac{c}{d}\left[N\pi + (n + m + 1)\cos\sqrt{u_1 u_2}\right] \qquad \text{[IV-10b]}$$

V. PLANE WAVE PROPAGATION IN A LINEAR, HOMOGENEOUS, ANISOTROPIC DIELECTRIC MEDIA ("CRYSTAL OPTICS"):

Our objective here is to formulate a general approach to the subject of wave propagation in anisotropic dielectrics which makes use of ideas familiar from other branches of mathematical physics -- *viz*. the "eigenvalue problem.".[18] For reasons that will soon become abundantly clear, treatments of "crystal optics" focus on the behavior of the dielectric displacement vector, $\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega)$ rather than on the electric field vector.[19] For non-magnetic dielectrics the components of the dielectric displacement are usefully represented as the Cartesian coordinates coordinates of figure called the "ellipsoid of wave normals," the "optical indicatix," the "index ellipsoid" or the "reciprocal ellipsoid."

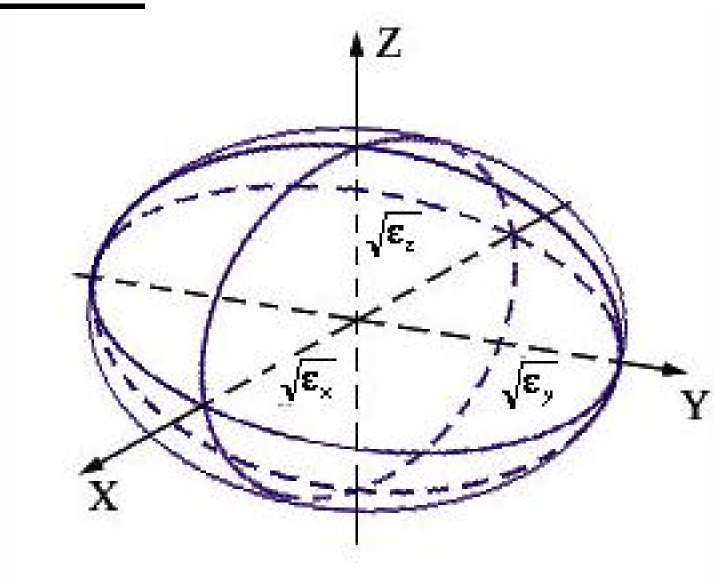
Since the stored electrical energy is given by

$$U_e = (1/2)\,\vec{\mathbf{D}}\;\vec{\mathbf{E}} = (1/2)\,\vec{\mathbf{D}}\;\vec{\varepsilon}^{\,-1}\;\vec{\mathbf{D}}$$

we can write in the *principle axis system*

$$\frac{\left(\dfrac{D_x}{\sqrt{2U_e}}\right)^2}{\varepsilon_x} + \frac{\left(\dfrac{D_y}{\sqrt{2U_e}}\right)^2}{\varepsilon_y} + \frac{\left(\dfrac{D_z}{\sqrt{2U_e}}\right)^2}{\varepsilon_z} = 1$$

where $\{\varepsilon_x, \varepsilon_y, \varepsilon_z\}$ are *principle axis* values of the dielectric constant tensor.



18 Kaiser S. Kunz in 1977 presented a similar treatment of this problem in a paper entitiled "Treatment of optical propagation in crystals using projection dyadics," Am. J. Phys., **Vol. 45**, 1977, pp. 267-269.

19 Perhaps the most authoritative treatment of "Crystal Optics" is found in Max Born and Emil Wolf, *Principle of Optics*, Pergamon Press (1986), Chapter 14.

Now let us first combine Equations [I-8a] and [I-8b] to obtain a generalized Helmholz equation for a homogeneous anisotropic dielectric

$$\vec{\;} \times \left(\vec{\;} \times \left[\vec{\varepsilon}^{-1}(\omega) \;\; \vec{\mathbf{D}}(\vec{\mathbf{r}},\omega) \right] \right) = \omega^2 \mu_o \; \vec{\mathbf{D}}(\vec{\mathbf{r}},\omega) \qquad \text{[V-1]}$$

In general, we would hope to be able to find a set of ***eigenmodes*** of the homogeneous problem that would satisfy the scalar eigenequation

$$^2 \; \mathrm{D}^{(\sigma)}(\vec{\mathbf{r}},\omega) + n_{(\sigma)}^2 \, k_0^2 \; \mathrm{D}^{(\sigma)}(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[V-2]}$$

where $n_{(\sigma)}$ is the effective index of refraction of the σ-th eigenmode. We can be quite specific for the case of **plane wave eigenmodes** where we suppose that all fields have a spatial dependence $\exp(\mp i \, \vec{\mathbf{k}} \;\; \vec{\mathbf{r}})$. From Equation [I-8c] we see that $\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega)$ must, in general, be orthogonal to $\vec{\mathbf{k}}$ so that we may write

$$\vec{\mathbf{D}}\,(\vec{\mathbf{k}},\omega) = \mathrm{D}^{(1)}(\vec{\mathbf{k}},\omega)\; \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}},\omega) + \mathrm{D}^{(2)}(\vec{\mathbf{k}},\omega)\; \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}},\omega) \qquad \text{[V-3]}$$

where the $\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})$'s are polarization unit vectors which are orthogonal to $\breve{\mathbf{k}}$, the unit vector parallel to $\vec{\mathbf{k}}$. Since the components of $\vec{\varepsilon}$ in the general case may be complex -- *e.g.,* in the case of magneto-optical media -- the eigenmodes may be polarized along "complex directions" -- *e.g.,* “screw axes” defining the sense of circular polarization -- and we must use great care in all vector manipulations. Thus, we define a set of adjoint or conjugate unit vectors by means of the relationships

$$\breve{\mathbf{k}} \;\; \hat{\mathbf{k}} = 1; \qquad \breve{\mathbf{k}} \;\; \hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}}) = 0; \qquad \breve{\mathbf{t}}^{(\sigma)}(\hat{\mathbf{k}}) \;\; \hat{\mathbf{k}} = 0; \qquad \breve{\mathbf{t}}^{(\sigma)}(\hat{\mathbf{k}}) \;\; \hat{\mathbf{t}}^{(\sigma')}(\breve{\mathbf{k}}) = \delta_{\sigma\sigma'} \qquad \text{[V-4]}$$

Thus, with $\vec{\mathbf{k}} = \mathrm{k}\,\hat{\mathbf{k}} = \mathrm{k}^* \; \breve{\mathbf{k}}$ (so that $\vec{\mathbf{k}} \;\; \vec{\mathbf{k}} = |\mathrm{k}|^2$) the representation for $\vec{\mathbf{D}}(\vec{\mathbf{k}})$ in Equation [V-3] automatically satisfies Equation. [I-8c]. A key problem is the representation of the ubiquitous vector operation

$$\overleftrightarrow{\ }(\vec{\mathbf{r}})\ \vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\ }\times\left(\vec{\ }\times\vec{\mathbf{F}}(\vec{\mathbf{r}})\right) = \vec{\ }\vec{\ }\ \vec{\mathbf{F}}(\vec{\mathbf{r}}) - {}^2\vec{\mathbf{F}}(\vec{\mathbf{r}}) \qquad [\text{ V-5 }]$$

which appears in Equation [V-1]. For plane wave representation of the operator becomes

$$\overleftrightarrow{\ }(\vec{\mathbf{k}}) \equiv \left(\vec{\mathbf{k}}\ \ \vec{\mathbf{k}}\right)\overleftrightarrow{\mathbf{1}} - \vec{\mathbf{k}}\,\vec{\mathbf{k}} = |\mathrm{k}|^2\left\{\overleftrightarrow{\mathbf{1}} - \breve{\mathbf{k}}\,\hat{\mathbf{k}}\right\}. \qquad [\text{ V-6 }]$$

Since

$$\overleftrightarrow{\mathbf{1}} = \hat{\mathbf{k}}\ \breve{\mathbf{k}} + \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}})\ \breve{\mathbf{t}}^{(1)}(\hat{\mathbf{k}}) + \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\ \breve{\mathbf{t}}^{(2)}(\hat{\mathbf{k}}) \qquad [\text{ V-7 }]$$

the plane wave representation of the operator simplifies to

$$\overleftrightarrow{\ }(\vec{\mathbf{k}}) = |\mathrm{k}|^2\left\{\hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}})\ \breve{\mathbf{t}}^{(1)}(\hat{\mathbf{k}}) + \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\ \breve{\mathbf{t}}^{(2)}(\hat{\mathbf{k}})\right\} \qquad [\text{ V-8}]$$

Using this representation of the $\overleftrightarrow{\ }(\vec{\mathbf{k}})$ operator and the representation for $\vec{\mathbf{D}}(\vec{\mathbf{k}})$ in Equation. [V-1], the generalized Helmholz equation -- *i.e.* Equation [V-1] -- can be written

$$\begin{aligned}
&|\mathrm{k}|^2\{\mathrm{D}^{(1)}(\vec{\mathbf{k}})\ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}})\left[\breve{\mathbf{t}}^{(1)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}})\right] + \mathrm{D}^{(1)}(\vec{\mathbf{k}})\ \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\left[\breve{\mathbf{t}}^{(2)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}})\right] + \\
&\mathrm{D}^{(2)}(\vec{\mathbf{k}})\ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}})\left[\breve{\mathbf{t}}^{(1)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\right] + \mathrm{D}^{(2)}(\vec{\mathbf{k}})\ \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\left[\breve{\mathbf{t}}^{(2)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\right]\} \qquad [\text{ V-9 }] \\
&= \omega^2\mu_0\left\{\mathrm{D}^{(1)}(\vec{\mathbf{k}})\ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}}) + \mathrm{D}^{(2)}(\vec{\mathbf{k}})\ \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\right\}
\end{aligned}$$

Crucial point: To obtain an eigenvalue equation we need to choose the eigenvectors $\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})$ so that

$$\breve{\mathbf{t}}^{(1)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}}) = \breve{\mathbf{t}}^{(2)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}}) \equiv 0 \qquad [\text{ V-10 }]$$

If we can find eigenvectors defined in this way, then Equation [I-9] becomes an eigenvalue equation with eigenvalues -- *i.e.*, the inverse refractive indices -- given by

$$\left[n_1(\vec{\mathbf{k}})\right]^{-2} = \mathrm{k}_0^2 \big/ |\mathrm{k}|^2 = \varepsilon_0\ \breve{\mathbf{t}}^{(1)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}}) \qquad [\text{ V-11a }]$$

$$\left[n_2(\vec{\mathbf{k}})\right]^{-2} = \mathrm{k}_0^2 \big/ |\mathrm{k}|^2 = \varepsilon_0 \; \breve{\mathbf{t}}^{(2)}(\widehat{\mathbf{k}}) \;\; \overleftrightarrow{\varepsilon}^{-1} \;\; \widehat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}}) \qquad [\text{ V-11b }]$$

These results -- *i.e.,* Equations [I-10] and [I-11] -- are a complete formal solution of the problem. However, they are difficult to apply in the general case and an addition relationship -- *viz.,* the Fresnel equation of wave normals -- is found to be extremely useful as the starting point for actual computations. For plane waves, Equation [V-1] can be rewritten as

$$\left[|\mathrm{k}|^2 \; \overleftrightarrow{\mathbf{1}} - \left(\mathrm{k}_0^2/\varepsilon_0\right) \overleftrightarrow{\varepsilon}\right] \;\; \vec{\mathbf{E}}\left(\vec{\mathbf{k}}\right) = \vec{\mathbf{k}}\,\vec{\mathbf{k}} \;\; \vec{\mathbf{E}}\left(\vec{\mathbf{k}}\right) \qquad [\text{ V-12 }]$$

Multiplying this equation through by $\vec{\mathbf{k}} \;\; \left[|\mathrm{k}|^2 \; \overleftrightarrow{\mathbf{1}} - \left(\mathrm{k}_0^2/\varepsilon_0\right) \overleftrightarrow{\varepsilon}\right]^{-1}$ we obtain

$$\vec{\mathbf{k}} \;\; \vec{\mathbf{E}}\left(\vec{\mathbf{k}}\right) = \left\{\vec{\mathbf{k}} \;\; \left[|\mathrm{k}|^2 \; \overleftrightarrow{\mathbf{1}} - \left(\mathrm{k}_0^2/\varepsilon_0\right) \overleftrightarrow{\varepsilon}\right]^{-1} \;\; \vec{\mathbf{k}}\right\} \; \vec{\mathbf{k}} \;\; \vec{\mathbf{E}}\left(\vec{\mathbf{k}}\right) \qquad [\text{ V-13a }]$$

or

$$1 = \vec{\mathbf{k}} \;\; \left[|\mathrm{k}|^2 \; \overleftrightarrow{\mathbf{1}} - \left(\mathrm{k}_0^2/\varepsilon_0\right) \overleftrightarrow{\varepsilon}\right]^{-1} \;\; \vec{\mathbf{k}} \qquad [\text{ V-13b }]$$

From this result we may develop two important relationships. Using the principal axes coordinates of the dielectric tensor, we can write

$$\left[|\mathrm{k}|^2 \; \overleftrightarrow{\mathbf{1}} - \left(\mathrm{k}_0^2/\varepsilon_0\right) \overleftrightarrow{\varepsilon}\right] = \sum_{\lambda=a}^{c} \left(|\mathrm{k}|^2 - \left(\mathrm{k}_0^2/\varepsilon_0\right) \varepsilon_{(\lambda)}\right) \widehat{\mathbf{a}}_\lambda \;\; \breve{\mathbf{a}}_\lambda \qquad [\text{ V-14a }]$$

and

$$\left[|\mathrm{k}|^2 \; \overleftrightarrow{\mathbf{1}} - \left(\mathrm{k}_0^2/\varepsilon_0\right) \overleftrightarrow{\varepsilon}\right]^{-1} = \sum_{\lambda=a}^{c} \left(|\mathrm{k}|^2 - \left(\mathrm{k}_0^2/\varepsilon_0\right) \varepsilon_{(\lambda)}\right)^{-1} \widehat{\mathbf{a}}_\lambda \;\; \breve{\mathbf{a}}_\lambda \qquad [\text{ V-14b }]$$

Therefore, Equation [I-14b] becomes

$$\frac{\left(\widehat{\mathbf{k}} \;\; \breve{\mathbf{a}}_a\right)\left(\breve{\mathbf{k}} \;\; \widehat{\mathbf{a}}_a\right)}{n^2 - n_a^2} + \frac{\left(\widehat{\mathbf{k}} \;\; \breve{\mathbf{a}}_b\right)\left(\breve{\mathbf{k}} \;\; \widehat{\mathbf{a}}_b\right)}{n^2 - n_b^2} + \frac{\left(\widehat{\mathbf{k}} \;\; \breve{\mathbf{a}}_c\right)\left(\breve{\mathbf{k}} \;\; \widehat{\mathbf{a}}_c\right)}{n^2 - n_c^2} = \frac{1}{n^2} \qquad [\text{ V-15a }]$$

where $n_\lambda^2 = \varepsilon_{(\lambda)}/\varepsilon_0$. Since $\sum_{\lambda=a}^{c} \hat{\mathbf{a}}_\lambda \, \breve{\mathbf{a}}_\lambda = \overleftrightarrow{\mathbf{1}}$ (or $\sum_{\lambda=a}^{c} \left(\breve{\mathbf{k}} \;\hat{\mathbf{a}}_\lambda\right)\left(\breve{\mathbf{a}}_\lambda \;\hat{\mathbf{k}}\right) = 1$) we may also write Equation [V-15a] as

$$\frac{\left(\hat{\mathbf{k}}\;\breve{\mathbf{a}}_a\right)\left(\breve{\mathbf{k}}\;\hat{\mathbf{a}}_a\right)}{\frac{1}{n^2}-\frac{1}{n_a^2}} + \frac{\left(\hat{\mathbf{k}}\;\breve{\mathbf{a}}_b\right)\left(\breve{\mathbf{k}}\;\hat{\mathbf{a}}_b\right)}{\frac{1}{n^2}-\frac{1}{n_b^2}} + \frac{\left(\hat{\mathbf{k}}\;\breve{\mathbf{a}}_c\right)\left(\breve{\mathbf{k}}\;\hat{\mathbf{a}}_c\right)}{\frac{1}{n^2}-\frac{1}{n_c^2}} = 0 \qquad \text{[V-15b]}$$

This latter expression is the famous ***Fresnel equation of wave normals***.[20]

APPLICATIONS OF THE FORMAL SOLUTION:

Uniaxial Dielectric Crystals:

For an optical material with uniaxial symmetry, the inverse dielectric tensor in the principal axes system must have the form [21]

$$\overline{\overline{\varepsilon}}^{-1} = \varepsilon^{-1}\left(\hat{\mathbf{a}}_a\,\hat{\mathbf{a}}_a + \hat{\mathbf{a}}_b\,\hat{\mathbf{a}}_b\right) + \varepsilon_{\|}^{-1}\,\hat{\mathbf{a}}_c\,\hat{\mathbf{a}}_c\ . \qquad \text{[V-16]}$$

Thus, Equation [I-19b] becomes

$$\left[\frac{1}{n^2}-\frac{1}{n^2}\right]\left\{\left[\frac{1}{n^2}-\frac{1}{n_{\|}^2}\right]\sin^2\vartheta + \left[\frac{1}{n^2}-\frac{1}{n^2}\right]\cos^2\vartheta\right\} = 0 \qquad \text{[V-17]}$$

so that

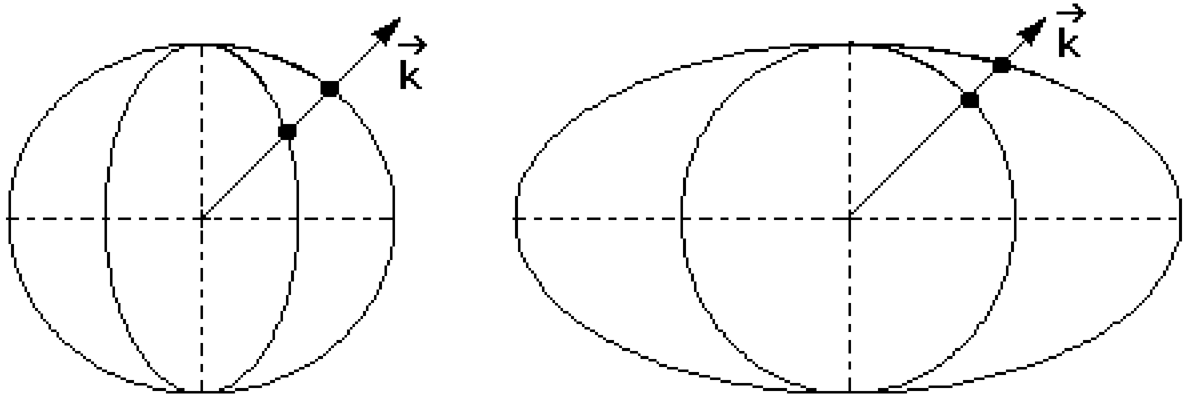
$$n_{\mathrm{o}}^{-2} = n^{-2}; \quad \text{and} \quad n_{\mathrm{e}}^{-2} = n^{-2}\cos^2\vartheta + n_{\|}^{-2}\sin^2\vartheta \qquad \text{[V-18]}$$

---

20 In its commonly used form, the Fresnel equation becomes $$\frac{\left(\hat{\mathbf{k}}\right)_x^2}{\frac{1}{n^2}-\frac{1}{n_x^2}} + \frac{\left(\hat{\mathbf{k}}\right)_y^2}{\frac{1}{n^2}-\frac{1}{n_y^2}} + \frac{\left(\hat{\mathbf{k}}\right)_z^2}{\frac{1}{n^2}-\frac{1}{n_z^2}} = 0$$

21 In this instance there is no need to trouble ourselves about conjugate unit vectors.

where $\hat{\mathbf{k}}\ \hat{\mathbf{a}}_c = \cos\vartheta$. The subscript **"o"** identifies the "ordinary" mode and the subscript **"e"** the "extraordinary" mode. These results are usually plotted as follows:



where the intersections of the $\vec{\mathbf{k}}$ vector yield the "ordinary" and "extraordinary" velocities of propagation for a given $\vec{\mathbf{k}}$.

Further, if we take

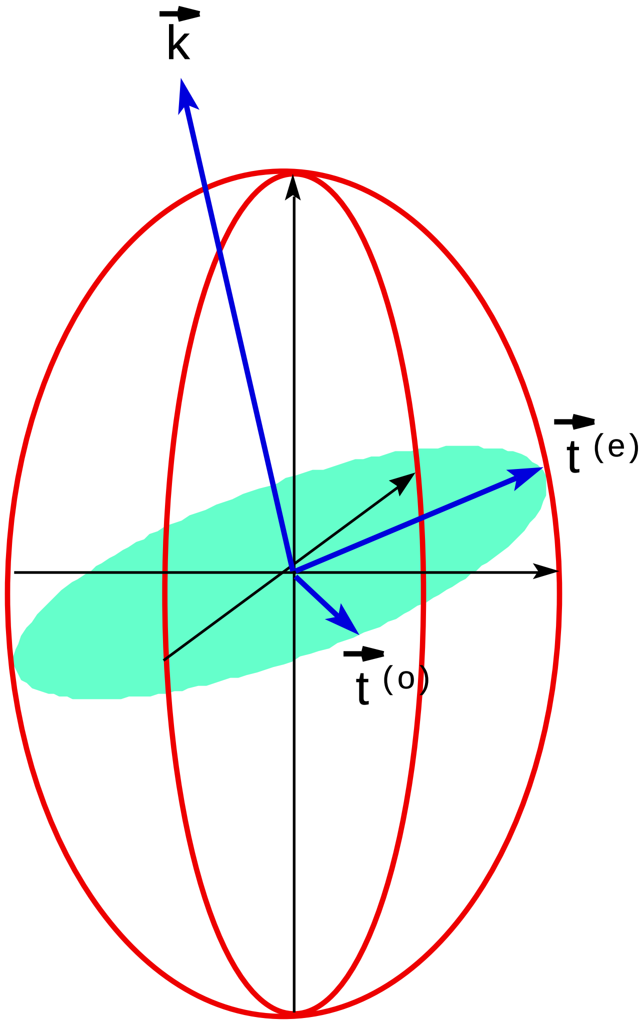
$$\hat{\mathbf{k}} = \sin\phi\ \sin\vartheta\ \hat{\mathbf{a}}_a + \cos\phi\ \sin\vartheta\ \hat{\mathbf{a}}_b + \cos\vartheta\ \hat{\mathbf{a}}_c \qquad \text{[V-19}$$

it is a *bagatelle* to show that

$$\widehat{\mathbf{t}}^{(o)}(\breve{\mathbf{k}}) = \breve{\mathbf{t}}^{(o)}(\widehat{\mathbf{k}}) = \cos\phi\ \hat{\mathbf{a}}_a - \sin\phi\ \hat{\mathbf{a}}_b \qquad \text{[V-20a]}$$

$$\widehat{\mathbf{t}}^{(e)}(\breve{\mathbf{k}}) = \breve{\mathbf{t}}^{(e)}(\widehat{\mathbf{k}}) = \cos\vartheta\ \left(\cos\phi\ \hat{\mathbf{a}}_a + \sin\phi\ \hat{\mathbf{a}}_b\right) - \sin\vartheta\ \hat{\mathbf{a}}_c \qquad \text{[V-20b]}$$

and that these quations are consistent with Equations [V-10] and [V-11].



Magneto-optical Media:

For a simple magneto-optical substance we may write the dielectric dyadic in the form [22]

[22] See, for example, Section 82 in L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press (1960).

$$\overleftrightarrow{\varepsilon}^{-1} = \left[\overline{\overline{\mathbf{1}}} + i\,\gamma\left(\hat{\mathbf{a}}_a\,\hat{\mathbf{a}}_b - \hat{\mathbf{a}}_b\,\hat{\mathbf{a}}_a\right)\right]\varepsilon^{-1} \qquad \text{[V-21]}$$

If we introduce the conjugate principal axes

$$\begin{aligned} \widehat{\mathbf{a}}_+ &= \breve{\mathbf{a}}_- = \left(1/\sqrt{2}\right)\left(\hat{\mathbf{a}}_a + i\,\hat{\mathbf{a}}_b\right) \\ \widehat{\mathbf{a}}_- &= \breve{\mathbf{a}}_+ = \left(1/\sqrt{2}\right)\left(\hat{\mathbf{a}}_a - i\,\hat{\mathbf{a}}_b\right) \\ \widehat{\mathbf{a}}_\parallel &= \breve{\mathbf{a}}_\parallel = \hat{\mathbf{a}}_c \end{aligned} \qquad \text{[V-22]}$$

we obtain the dielectric dyadic in the so called ***normal form*** -- *viz*.

$$\overleftrightarrow{\varepsilon}^{-1} = \left[\left(1-\gamma\right)\widehat{\mathbf{a}}_+\,\breve{\mathbf{a}}_+ + \left(1+\gamma\right)\widehat{\mathbf{a}}_-\,\breve{\mathbf{a}}_- + \widehat{\mathbf{a}}_\parallel\,\breve{\mathbf{a}}_\parallel\right]\varepsilon^{-1} \qquad \text{[V-23]}$$

Again from Equation [V-15b] it is trivial to show that

$$n_\pm^{-2} = \left[1 \pm \gamma\,\cos\vartheta\right]\varepsilon^{-1} \qquad \text{[V-24]}$$

Using the resolution of $\hat{\mathbf{k}}$ as given in Equation [V-4] we may show that

$$\widehat{\mathbf{t}}^{\pm}(\breve{\mathbf{k}}) = \left(\frac{i}{2}\right)\left\{\exp\left(-i\,\phi\right)\left[1 \mp \cos\vartheta\right]\widehat{\mathbf{a}}_+ - \exp\left(i\,\phi\right)\left[1 \pm \cos\vartheta\right]\widehat{\mathbf{a}}_- \pm \sqrt{2}\,\sin\vartheta\;\widehat{\mathbf{a}}_\parallel\right\} \qquad \text{[V-25a]}$$

and

$$\breve{\mathbf{t}}^{\pm}(\widehat{\mathbf{k}}) = \left(\frac{i}{2}\right)\left\{-\exp\left(i\,\phi\right)\left[1 \mp \cos\vartheta\right]\breve{\mathbf{a}}_+ + \exp\left(-i\,\phi\right)\left[1 \pm \cos\vartheta\right]\breve{\mathbf{a}}_- \mp \sqrt{2}\,\sin\vartheta\;\breve{\mathbf{a}}_\parallel\right\} \qquad \text{[V-25b]}$$

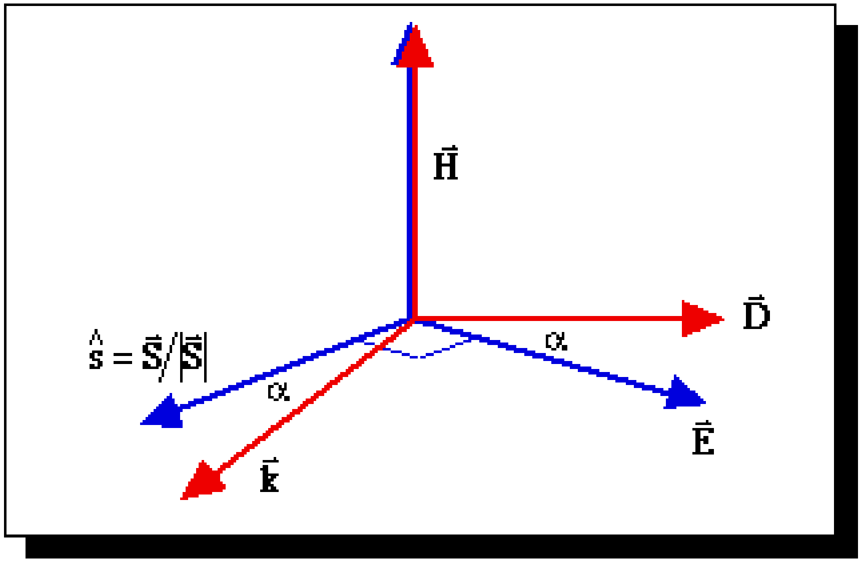
ENERGY FLOW IN ANISOTROPIC MEDIA

As previously noted, the content of Equations [I-10] and [I-11] represents in some sense a complete formal solution of the wave propagation problem. However, from a practical point of view it is essential to consider how energy propagates in anisotropic

media. To that end, we note that the time averaged Poynting vector associated with a given plane wave-- *viz*.

$$\vec{\mathbf{S}}\left(\vec{\mathbf{k}},\omega\right)=\frac{1}{2}\,\vec{\mathbf{E}}\left(\vec{\mathbf{k}},\omega\right)\times\vec{\mathbf{H}}^{*}\left(\vec{\mathbf{k}},\omega\right) \qquad \text{[V-26]}$$

propagates in a direction $\hat{\mathbf{s}}\left(\vec{\mathbf{k}},\omega\right)$ -- conventionally designated the “ray” direction -- which is orthogonal to both $\vec{\mathbf{E}}\left(\vec{\mathbf{k}},\omega\right)$ and $\vec{\mathbf{H}}\left(\vec{\mathbf{k}},\omega\right)$ as shown below.



The time averaged total stored energy is given by

$$\begin{aligned} U\left(\vec{\mathbf{k}},\omega\right) &= \frac{1}{4}\left[\vec{\mathbf{E}}\left(\vec{\mathbf{k}},\omega\right)\;\vec{\mathbf{D}}^{*}\left(\vec{\mathbf{k}},\omega\right)+\vec{\mathbf{B}}\left(\vec{\mathbf{k}},\omega\right)\;\vec{\mathbf{H}}^{*}\left(\vec{\mathbf{k}},\omega\right)\right] \\ &= \frac{\left|\vec{\mathbf{k}}\right|}{2\,\omega}\,\hat{\mathbf{k}}\;\;\vec{\mathbf{E}}\left(\vec{\mathbf{k}},\omega\right)\times\vec{\mathbf{H}}^{*}\left(\vec{\mathbf{k}},\omega\right)=\frac{\left|\vec{\mathbf{k}}\right|}{\omega}\,\hat{\mathbf{k}}\;\;\vec{\mathbf{S}}\left(\vec{\mathbf{k}},\omega\right) \end{aligned} \qquad \text{[V-27]}$$

and, thus, we see that the "ray" or "energy flow" velocity, $\mathrm{v}_{\mathrm{ray}}$, for a given $\hat{\mathbf{s}}\left(\vec{\mathbf{k}},\omega\right)$ is given by

$$\frac{1}{\mathrm{v}_{\mathrm{ray}}} = \frac{\left|\vec{\mathbf{k}}\right|}{\omega}\,\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}\left(\hat{\mathbf{k}},\omega\right) = \frac{1}{\mathrm{v}_{\mathrm{phase}}}\,\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}\left(\hat{\mathbf{k}},\omega\right) \qquad \text{[V-28]}$$

We write the time averaged Poynting vector associated with a given eigenmode as

$$\begin{aligned}\vec{\mathbf{S}}^{(\sigma)}\left(\vec{\mathbf{k}}\right) &= \frac{1}{2}\,\vec{\mathbf{E}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)\times\vec{\mathbf{H}}^{(\sigma)^*}\left(\vec{\mathbf{k}}\right) = \frac{1}{2\,\omega\,\mu_0}\,\vec{\mathbf{E}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)\times\left[\vec{\mathbf{k}}\times\vec{\mathbf{E}}^{(\sigma)^*}\left(\vec{\mathbf{k}}\right)\right] \\ &= \frac{1}{2\,\omega\,\mu_0}\left\{\vec{\mathbf{k}}\left|\vec{\mathbf{E}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)\right|^2 - \vec{\mathbf{E}}^{(\sigma)^*}\left(\vec{\mathbf{k}},\omega\right)\left[\vec{\mathbf{k}}\cdot\vec{\mathbf{E}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)\right]\right\}\end{aligned} \qquad \text{[V-29]}$$

where $\vec{\mathbf{E}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)$ is the electric field associated with the eigenmode. Using Equations [V-3], [V-10] and [V-11] this field can be expressed as

$$\begin{aligned}\vec{\mathbf{E}}^{(\sigma)}\left(\vec{\mathbf{k}}\right) &= \left\{\left[\breve{\mathbf{t}}^{(\sigma)}(\hat{\mathbf{k}})\cdot\overleftrightarrow{\varepsilon}^{-1}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\right]\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}}) + \left[\hat{\mathbf{k}}\cdot\overleftrightarrow{\varepsilon}^{-1}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\right]\hat{\mathbf{k}}\right\}\mathrm{D}^{(\sigma)}(\vec{\mathbf{k}}) \\ &= \frac{1}{\varepsilon_0\left[n_{(\sigma)}(\vec{\mathbf{k}})\right]^2}\left\{\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}}) + \varepsilon_0\left[n_{(\sigma)}(\vec{\mathbf{k}})\right]^2\left[\hat{\mathbf{k}}\cdot\overleftrightarrow{\varepsilon}^{-1}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\right]\hat{\mathbf{k}}\right\}\mathrm{D}^{(\sigma)}(\vec{\mathbf{k}}) \\ &= \frac{1}{\varepsilon_0\left[n_{(\sigma)}(\vec{\mathbf{k}})\right]^2}\left\{\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}}) + g^{(\sigma)}(\breve{\mathbf{k}})\,\hat{\mathbf{k}}\right\}\mathrm{D}^{(\sigma)}(\vec{\mathbf{k}})\end{aligned} \qquad \text{[V-30]}$$

where $g^{(\sigma)}(\breve{\mathbf{k}}) = \varepsilon_0\left[n_{(\sigma)}(\vec{\mathbf{k}})\right]^2\left[\hat{\mathbf{k}}\cdot\overleftrightarrow{\varepsilon}^{-1}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\right] = \dfrac{\left[\hat{\mathbf{k}}\cdot\overleftrightarrow{\varepsilon}^{-1}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\right]}{\left[\breve{\mathbf{t}}^{(\sigma)}(\hat{\mathbf{k}})\cdot\overleftrightarrow{\varepsilon}^{-1}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\right]}$. Using this parameterization, the modal Poynting vector can be expressed as

$$\vec{\mathbf{S}}^{(\sigma)}\left(\vec{\mathbf{k}}\right) = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{\left|\mathrm{D}^{(\sigma)}(\vec{\mathbf{k}})/\varepsilon_0\right|^2}{\left[n_{(\sigma)}(\vec{\mathbf{k}})\right]^3}\left\{\hat{\mathbf{k}} - \hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\,g^{(\sigma)}(\breve{\mathbf{k}})\right\} \qquad \text{[V-31]}$$

and the associated ray vector as

$$\hat{\mathbf{s}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)=\frac{\widehat{\mathbf{k}}-\widehat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\,g^{(\sigma)}(\breve{\mathbf{k}})}{\sqrt{\left|\widehat{\mathbf{k}}-\widehat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\,g^{(\sigma)}(\breve{\mathbf{k}})\right|}}=\frac{\widehat{\mathbf{k}}-\widehat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\,g^{(\sigma)}(\breve{\mathbf{k}})}{\sqrt{1+\left|g^{(\sigma)}(\breve{\mathbf{k}})\right|^{2}}} \qquad \text{[V-32]}$$

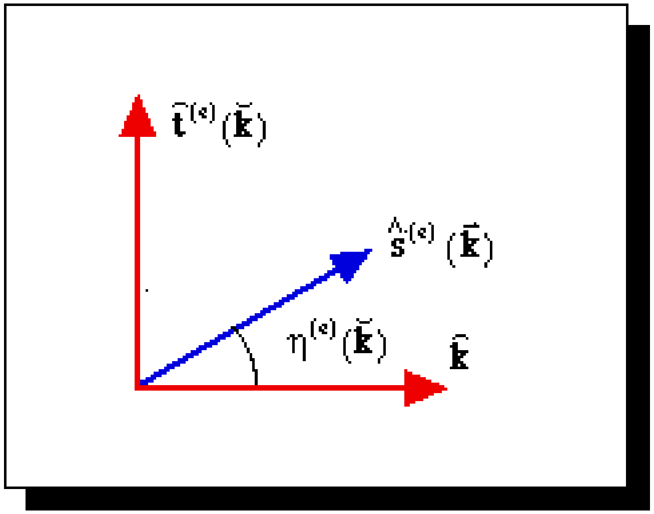
If we take $\widehat{\mathbf{k}}$ and $\widehat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})$ as reference directions, the ray vector, $\hat{\mathbf{s}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)$, lies in the plane containing $\widehat{\mathbf{k}}$ and $\widehat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})$ at an angle $\eta^{(\sigma)}(\breve{\mathbf{k}})=-\tan^{-1}\left[g^{(\sigma)}(\breve{\mathbf{k}})\right]$ with respect to the direction $\widehat{\mathbf{k}}$.

For an optical material with uniaxial symmetry, we may use Equations [V-16], [V-19] and [V-20] to evaluate $g^{(\sigma)}(\breve{\mathbf{k}})$. In particular, we may easily see that for the ordinary mode

$$\tan\eta^{(o)}(\breve{\mathbf{k}})=-g^{(o)}(\breve{\mathbf{k}})=0 \qquad \text{[V-33a]}$$

or $\hat{\mathbf{s}}^{(o)}\left(\vec{\mathbf{k}}\right)=\widehat{\mathbf{k}}$ and for the extraordinary mode

$$\tan\eta^{(e)}(\breve{\mathbf{k}})=-g^{(e)}(\breve{\mathbf{k}})=\frac{\sin\vartheta\,\cos\vartheta\,\left(\varepsilon^{-1}-\varepsilon_{\|}^{-1}\right)}{\varepsilon^{-1}\cos^{2}\vartheta+\varepsilon_{\|}^{-1}\sin^{2}\vartheta}\,. \qquad \text{[V-33b]}$$


$\hat{t}^{(e)}(\vec{k})$
$\hat{s}^{(e)}(\vec{k})$
$\eta^{(e)}(\vec{k})$
$\hat{k}$

VI. DESCRIPTIONS OF POLARIZED LIGHT[22]

Consider a **totally coherent** wave propagating in the positive direction

$$\mathbf{E_x} = \mathbf{E}_x^0 \cos(\omega t + \delta_x) = \mathbf{E}_x^0 \left[\exp\left[i(\omega t + \delta_x)\right] + c.c.\right] \qquad \text{[VI-1a]}$$

$$\mathbf{E}_y = \mathbf{E}_y^0 \cos(\omega t + \delta_y) = \mathbf{E}_y^0 \left[\exp\left[i(\omega t + \delta_y)\right] + c.c.\right] \qquad \text{[VI-1b]}$$

For later reference, we note that the *full* (non-normalized) Jones vector representation [23] of this field is given by

$$\vec{\mathbf{J}} = \begin{bmatrix} \mathbf{E}_x^0 \exp i\delta_x \\ \mathbf{E}_y^0 \exp i\delta_y \end{bmatrix} \qquad \text{[VI-2]}$$

We can be easily shown that

$$\cos\omega t = \left[\sin(\delta_y - \delta_x)\right]^{-1} \left[\frac{\mathbf{E_x}}{\mathbf{E}_x^0}\sin\delta_y - \frac{\mathbf{E}_y}{\mathbf{E}_y^0}\sin\delta_x\right] \qquad \text{[VI-3a]}$$

$$\sin\omega t = \left[\sin(\delta_y - \delta_x)\right]^{-1} \left[\frac{\mathbf{E_x}}{\mathbf{E}_x^0}\cos\delta_y - \frac{\mathbf{E}_y}{\mathbf{E}_y^0}\cos\delta_x\right] \qquad \text{[VI-3b]}$$

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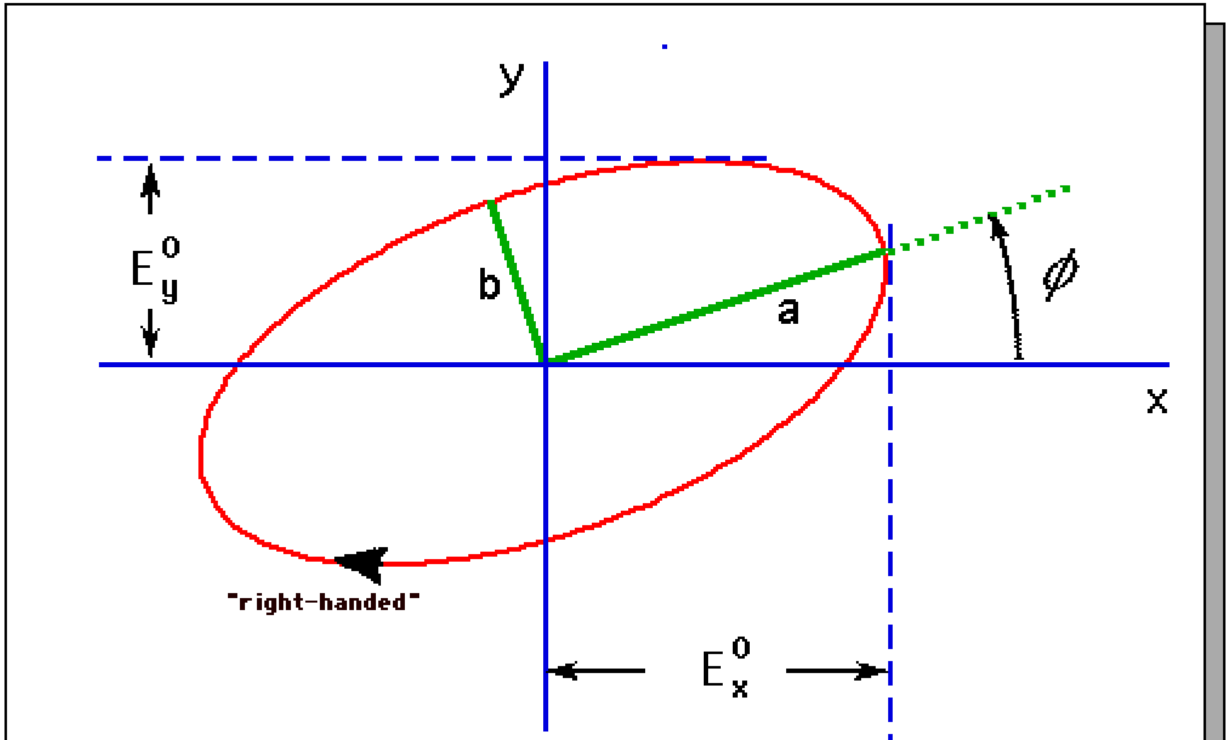
22 The best references on this subject are the following: 1.) William A. Shurcliff, *Polarized Light: Production and Use*, Harvard University Press (1962); 2.) D. Clarke and J.F. Grainger, *Polarized Light and Optical Measurement*, Pergamon Press (1971); 3.) Max Born and Emil Wolf, *Principles of Optics*, Pergamon Press (Particularly Section 1.4).

23 R. Clarke Jones, "New calculus for the treatment of optical systems. I. Description and discussion of calculus," *J. Opt. Soc. Amer. 31*, 488 (1941). To quote Shurcliff,

> "The Jones vector, … describes a polarized beam with the maximum algebraic brevity, and is eminently suited to the solving of problems involving beams whose phase relations phase relations are important. …"

Thus the locus of time sequence of fields in a plane perpendicular to the direction of propagtion follows an ellipse -- *viz*

$$\left(\frac{\mathbf{E}_x}{\mathbf{E}_x^0}\right)^2+\left(\frac{\mathbf{E}_y}{\mathbf{E}_y^0}\right)^2-2\left(\frac{\mathbf{E}_x}{\mathbf{E}_x^0}\right)\left(\frac{\mathbf{E}_y}{\mathbf{E}_y^0}\right)\cos\left(\delta_y-\delta_x\right)=\sin^2\left(\delta_y-\delta_x\right) \qquad \text{[VI-4]}$$



The "plane of polarization" with propagation out of the plane

For an electric field vector "seen" to be rotating in a clockwise direction by an observer receiving the radiation (*i.e.*, $\pi > \left(\delta_y-\delta_x\right) > 0$), the polarization is said to be *right-handed*. For rotation in the anticlockwise sense (*i.e.*, $0 > \left(\delta_y-\delta_x\right) > -\pi$), the polarization is said to be *right-handed*. Although this is a complete decription of the coherent field, it is not a

convenient one. It is useful to transform this equation to its principal axes form by the following transformation

$$\mathbf{E_x} = \mathbf{E}_\xi \cos\varphi - \mathbf{E}_\zeta \sin\varphi \qquad \text{[VI-5a]}$$

$$\mathbf{E}_y = \mathbf{E}_\zeta \cos\varphi + \mathbf{E}_\xi \sin\varphi \qquad \text{[VI-5b]}$$

where we choose φ so that

$$\left(\frac{\mathbf{E}_\xi}{a}\right)^2 + \left(\frac{\mathbf{E}_\zeta}{b}\right)^2 = 1 \qquad \text{[VI-6a]}$$

$$\mathbf{E}_\xi = a\cos\left(\omega t + \delta_0\right) \qquad \text{[VI-6b]}$$

$$\mathbf{E}_\zeta = \pm\, b\sin\left(\omega t + \delta_0\right) \qquad \text{[VI-6c]}$$

Where the upper and lower signs are for, respectively, *right-handed* and *left-handed* polarizations. After quite a bit of algebraic manipulation, we find a more elegant and convenient description of polarization in terms of the following set of relationships

$$\left(\mathbf{E}_x^0\right)^2 + \left(\mathbf{E}_y^0\right)^2 = a^2 + b^2 \qquad \text{[VI-7a]}$$

$$\pm\, a\, b = \left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\sin\left(\delta_y - \delta_x\right) = \left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\sin\delta \qquad \text{[VI-7b]}$$

$$\tan 2\varphi = \frac{2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\cos\left(\delta_y - \delta_x\right)}{\left(\mathbf{E}_x^0\right)^2 - \left(\mathbf{E}_y^0\right)^2} = \frac{2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\cos\delta}{\left(\mathbf{E}_x^0\right)^2 - \left(\mathbf{E}_y^0\right)^2} \qquad \text{[VI-7c]}$$

$$\pm\frac{2\,a\,b}{a^2 + b^2} = \frac{2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\sin\left(\delta_y - \delta_x\right)}{\left(\mathbf{E}_x^0\right)^2 + \left(\mathbf{E}_y^0\right)^2} = \frac{2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\sin\delta}{\left(\mathbf{E}_x^0\right)^2 + \left(\mathbf{E}_y^0\right)^2} \qquad \text{[VI-7d]}$$

These relations are simplified if we introduce the two auxiliary angles

$$\tan\alpha = \mathbf{E}_y^0 / \mathbf{E}_x^0 , \qquad \text{[VI-8a]}$$

and

$$\tan\theta = \pm\, b / a , \qquad \text{[VI-8b]}$$

(Note: $\frac{b-a}{b}$ specifies the, so called, *ellipticity* of the vibrational ellipse.). In terms of these auxiliary, we may then write

$$\left(\mathbf{E}_x^0\right)^2 + \left(\mathbf{E}_y^0\right)^2 = a^2 + b^2 \qquad \text{[VI-9a]}$$

$$\tan 2\varphi = \frac{2\tan\alpha}{1-\tan^2\alpha}\cos\delta = \left(\tan 2\alpha\right)\cos\delta \qquad \text{[VI-9b]}$$

$$\sin 2\theta = \left(\sin 2\alpha\right)\sin\delta \qquad \text{[VI-9c]}$$

Probably, the most powerful representation of polarization is found in the famous Stokes vector or parameters -- *viz.*[24]

$$I = \left(\mathbf{E}_x^0\right)^2 + \left(\mathbf{E}_y^0\right)^2 \qquad \text{[VI-10a]}$$

$$M = \left(\mathbf{E}_x^0\right)^2 - \left(\mathbf{E}_y^0\right)^2 \qquad \text{[VI-10b]}$$

$$C = 2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\cos\left(\delta_y - \delta_x\right) = 2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\cos\delta \qquad \text{[VI-10c]}$$

$$S = 2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\sin\left(\delta_y - \delta_x\right) = 2\left(\mathbf{E}_x^0\right)\left(\mathbf{E}_y^0\right)\sin\delta \qquad \text{[VI-10d]}$$

so that a polarized field may be represented by the vector

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24 G. G. Stokes, "On the composition and resolution of streams of polarized light from different sources," *Trans. Cambridge Phil. Soc. 9*, 399 (1852).

$$\begin{bmatrix} I \\ M \\ C \\ S \end{bmatrix} \qquad \text{[VI-11a]}$$

or its transpose $\{I \quad M \quad C \quad S\}$ [VI-11b]

These vector components give a complete geometric description of the vibrational ellipse -- *viz*.

I	Size
$C/M = \tan 2\varphi$	Azimuth
$\lvert S\rvert/I = \sin 2\theta = \dfrac{2b/a}{1+(b/a)^2}$	Shape
Sign of S	Handedness

As can see from Equations [VI-10], for a completely polarized field only three of these parameters or vector components are independent, since

$$I^2 = M^2 + C^2 + S^2 \qquad \text{[VI-9]}$$

and $\{M, C, S\}$ can be interpreted as the Cartesian coordinates of a sphere of radius I -- the Poincaré sphere.[25] Thus, from Equations [VI-8] and [VI-10] we obtain the coordinates of the Poincaré sphere or representation as

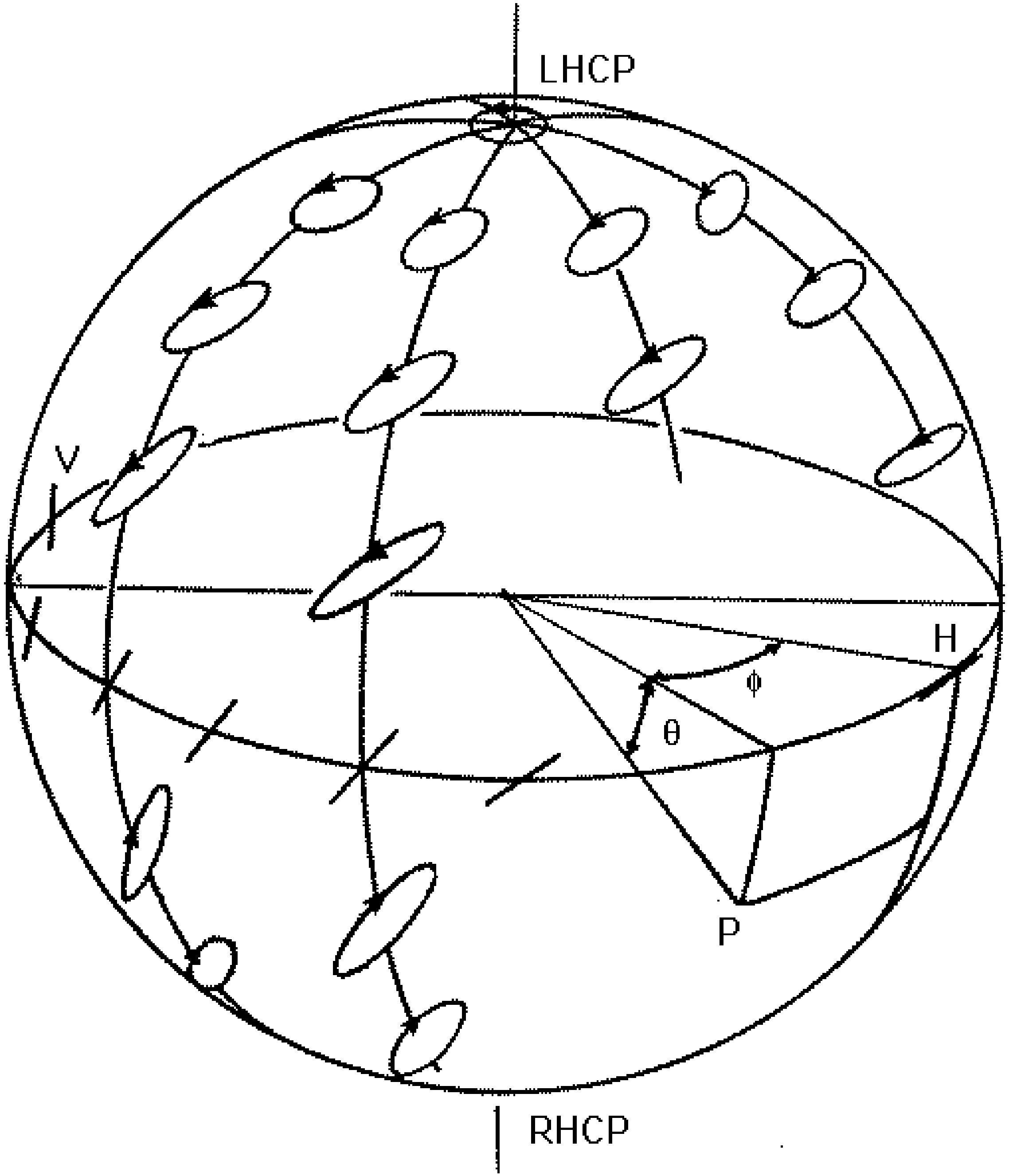
[25] H. Poincaré, *Théorie Mathématique de la Lumière*, Vol. 2, (1892) Chap. 12.

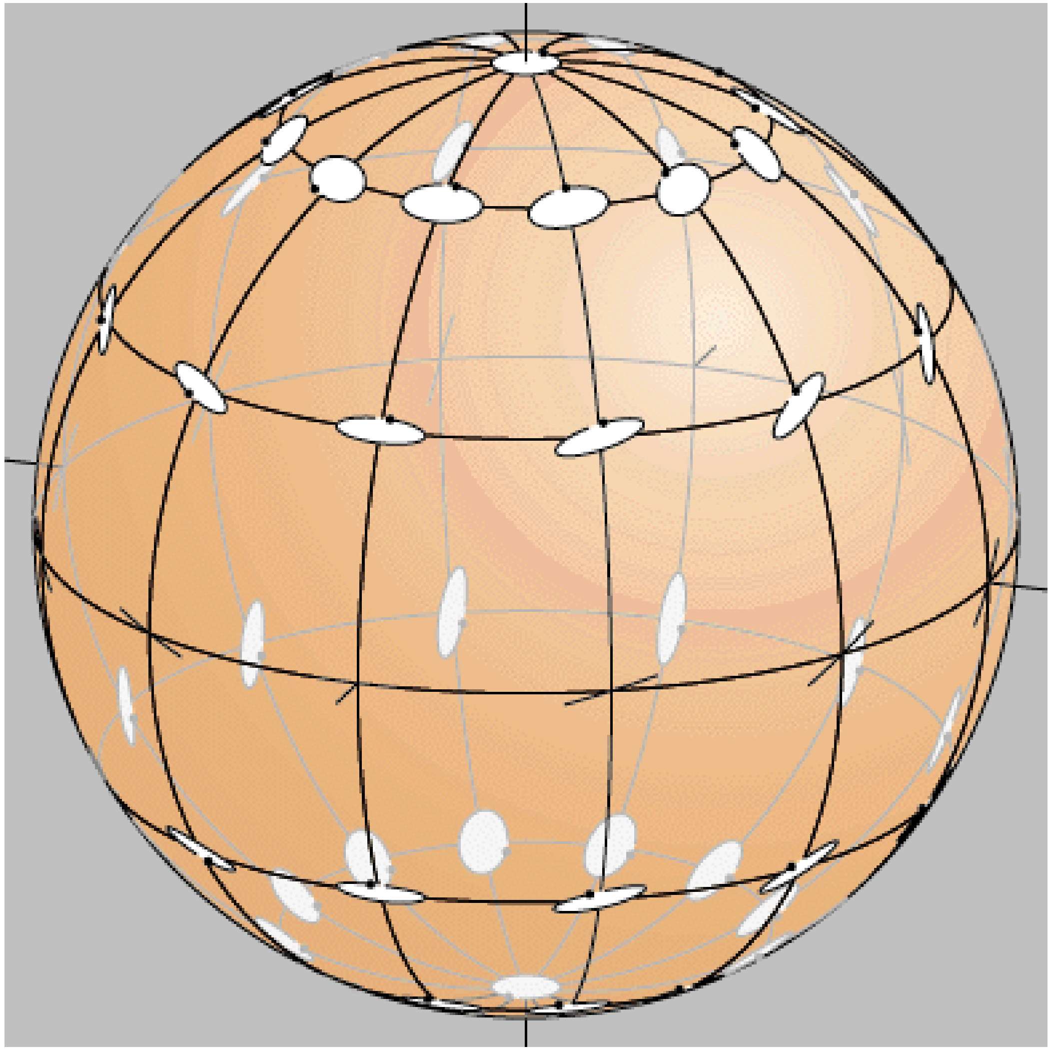
$$I^2 = M^2 + C^2 + S^2 \qquad \text{[VI-10a]}$$

$$M = I \cos 2\theta \, \cos 2\varphi \qquad \text{[VI-10b]}$$

$$C = I \cos 2\theta \sin 2\varphi \qquad \text{[VI-10b]}$$

$$S = I \sin 2\theta \qquad \text{[VI-10b]}$$


LHCP
V
H
φ
θ
P
RHCP



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VII. NONLINEAR OPTICS -- CLASSICAL PICTURE:

AN EXTENDED PHENOMENOLOGICAL MODEL OF POLARIZATION:

As an introduction to the subject of nonlinear optical phenomena, we write, in the spirit of Equation [I-4], the most general form of higher order terms in the phenomenological electric field expansion of the polarization density (which may then be inserted in Equations [I-3]) as

$$
\begin{aligned}
\mathbf{P}_{\alpha}^{(NL)}(\vec{\mathbf{r}},t) = {} & \varepsilon_0 \sum_{\beta\gamma} \int_{\vec{\mathbf{r}}_1}\int_{\vec{\mathbf{r}}_2}\int_{t_1}\int_{t_2} d\vec{\mathbf{r}}_1\, dt_1\, d\vec{\mathbf{r}}_2\, dt_2\, \chi^{(2)}_{\alpha\beta\gamma}\left(\vec{\mathbf{r}}-\vec{\mathbf{r}}_1, t-t_1; \vec{\mathbf{r}}-\vec{\mathbf{r}}_2, t-t_2\right) \mathbf{E}_{\beta}\left(\vec{\mathbf{r}}_1, t_1\right) \mathbf{E}_{\gamma}\left(\vec{\mathbf{r}}_2, t_2\right) \\
& + \varepsilon_0 \sum_{\beta\gamma\delta} \int_{\vec{\mathbf{r}}_1}\int_{\vec{\mathbf{r}}_2}\int_{\vec{\mathbf{r}}_3}\int_{t_1}\int_{t_2}\int_{t_3} d\vec{\mathbf{r}}_1\, dt_1\, d\vec{\mathbf{r}}_2\, dt_2\, d\vec{\mathbf{r}}_3\, dt_3\, \chi^{(3)}_{\alpha\beta\gamma\delta}\left(\vec{\mathbf{r}}-\vec{\mathbf{r}}_1, t-t_1; \vec{\mathbf{r}}-\vec{\mathbf{r}}_2, t-t_2; \vec{\mathbf{r}}-\vec{\mathbf{r}}_3, t-t_3\right) \\
& \times \mathbf{E}_{\beta}\left(\vec{\mathbf{r}}_1, t_1\right) \mathbf{E}_{\gamma}\left(\vec{\mathbf{r}}_2, t_2\right) \mathbf{E}_{\delta}\left(\vec{\mathbf{r}}_3, t_3\right) + \cdots .
\end{aligned}
\qquad \text{[VII-1]}
$$

The wave vector and frequency dependent **second and third order susceptibilities** are then defined as

$$
\begin{aligned}
\chi^{(2)}_{\alpha\beta\gamma}\left(\vec{\mathbf{k}}_1, \omega_1; \vec{\mathbf{k}}_2, \omega_2\right) = {} & \int_{\mathbf{R}_1}\int_{\mathbf{R}_2}\int_{\tau_1}\int_{\tau_2} d\vec{\mathbf{R}}_1 d\tau_1\, d\vec{\mathbf{R}}_2 d\tau_2\, \exp\left[-i\, \vec{\mathbf{k}}_1 \cdot \vec{\mathbf{R}}_1\right] \exp\left[+i\omega_1 \tau_1\right] \\
& \times \exp\left[-i\, \vec{\mathbf{k}}_2 \cdot \vec{\mathbf{R}}_2\right] \exp\left[+i\omega_2 \tau_2\right] \chi^{(2)}_{\alpha\beta\gamma}\left(\vec{\mathbf{R}}_1, \tau_1; \vec{\mathbf{R}}_2, \tau_2\right)
\end{aligned}
\qquad \text{[VII-2a]}
$$

and

$$
\begin{aligned}
\chi^{(3)}_{\alpha\beta\gamma\delta}\left(\vec{\mathbf{k}}_1, \omega_1; \vec{\mathbf{k}}_2, \omega_2; \vec{\mathbf{k}}_3, \omega_3\right) = {} & \int_{\mathbf{R}_1}\int_{\mathbf{R}_2}\int_{\mathbf{R}_3}\int_{\tau_1}\int_{\tau_2}\int_{\tau_3} d\vec{\mathbf{R}}_1\, d\tau_1\, d\vec{\mathbf{R}}_2\, d\tau_2\, d\vec{\mathbf{R}}_3\, d\tau_3\, \exp\left[-i\, \vec{\mathbf{k}}_1 \cdot \vec{\mathbf{R}}_1\right] \exp\left[+i\omega_1 \tau_1\right] \\
& \times \exp\left[-i\, \vec{\mathbf{k}}_2 \cdot \vec{\mathbf{R}}_2\right] \exp\left[+i\omega_2 \tau_2\right] \exp\left[-i\, \vec{\mathbf{k}}_3 \cdot \vec{\mathbf{R}}_3\right] \exp\left[+i\omega_3 \tau_3\right] \\
& \times \chi^{(3)}_{\alpha\beta\gamma\delta}\left(\vec{\mathbf{R}}_1, \tau_1; \vec{\mathbf{R}}_2, \tau_2; \vec{\mathbf{R}}_3, \tau_3\right)
\end{aligned}
\quad . \qquad \text{[VII-2b]}
$$

Thus, we may write quite generally [26]

$$\begin{aligned} P^{(NL)}_{\alpha}(\vec{\mathbf{r}},t) = {} & \varepsilon_0 \sum_{\beta\gamma} \int_{\mathbf{k}_1}\int_{\mathbf{k}_2}\int_{\omega_1}\int_{\omega_2} d\vec{\mathbf{k}}_1\, d\omega_1\, d\vec{\mathbf{k}}_2\, d\omega_2 \, \exp\left[i\left(\vec{\mathbf{k}}_1+\vec{\mathbf{k}}_2\right)\cdot\vec{\mathbf{r}}\right]\exp\left[-i\left(\omega_1+\omega_2\right)t\right] \\ & \times \chi^{(2)}_{\alpha\beta\gamma}\left(\vec{\mathbf{k}}_1,\omega_1;\vec{\mathbf{k}}_2,\omega_2\right)\mathbf{E}_\beta\left(\vec{\mathbf{k}}_1,\omega_1\right)\mathbf{E}_\gamma\left(\vec{\mathbf{k}}_2,\omega_2\right) \\ & + \varepsilon_0 \sum_{\beta\gamma\delta} \int_{\mathbf{k}_1}\int_{\mathbf{k}_2}\int_{\mathbf{k}_3}\int_{\omega_1}\int_{\omega_2}\int_{\omega_3} d\vec{\mathbf{k}}_1\, d\omega_1\, d\vec{\mathbf{k}}_2\, d\omega_2\, d\vec{\mathbf{k}}_3\, d\omega_3 \, \exp\left[i\left(\vec{\mathbf{k}}_1+\vec{\mathbf{k}}_2+\vec{\mathbf{k}}_3\right)\cdot\vec{\mathbf{r}}\right]\exp\left[-i\left(\omega_1+\omega_2+\omega_3\right)t\right] \\ & \times \chi^{(3)}_{\alpha\beta\gamma\delta}\left(\vec{\mathbf{k}}_1,\omega_1;\vec{\mathbf{k}}_2,\omega_2;\vec{\mathbf{k}}_3,\omega_3\right)\mathbf{E}_\beta\left(\vec{\mathbf{k}}_1,\omega_1\right)\mathbf{E}_\gamma\left(\vec{\mathbf{k}}_2,\omega_2\right)\mathbf{E}_\delta\left(\vec{\mathbf{k}}_3,\omega_3\right)+\cdots. \end{aligned} \qquad \text{[VII-3]}$$

A SIMPLE CLASSICAL MODEL OF NONLINEAR OPTICAL RESPONSE

A simple Lorentz-Dude model is often used in the literature as a valuable guide to the understanding of the frequency behavior of the nonlinear dielectric response.[27] We assume that the potential energy of a one-dimensional nonlinear (anharmonic) oscillator may be written

$$V(x) = V_2(x) + V_3(x) + V_4(x) + \cdots = \frac{1}{2} M\,\omega_o^2\, x^2 + \frac{1}{3} M a\, x^3 + \frac{1}{4} M b\, x^4 + \cdots \qquad \text{[VII-4]}$$

(See figures on next page)

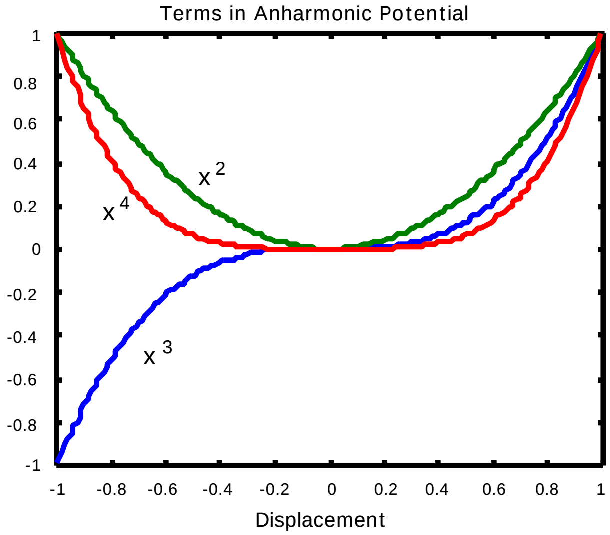
Thus, the equation of motion of a particle moving in that potential becomes

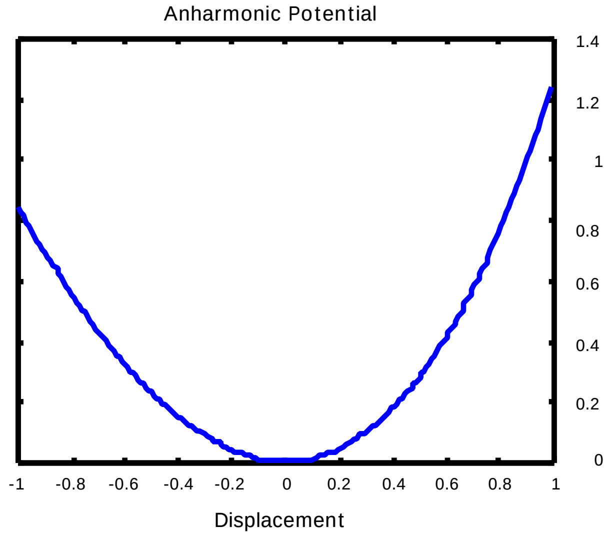
$$\ddot{x} + \Gamma\,\dot{x} + \omega_o^2\, x + a\, x^2 + b\, x^3 + \cdots = \frac{q}{M}\,\mathrm{E}(t) \qquad \text{[VII-5]}$$

---

[26] $\chi^{(2)}_{\alpha\beta\gamma}$ must vanish for any material that is invariant under inversion, since both $\vec{\mathbf{P}}$ and $\vec{\mathbf{E}}$ are vectors, and are thus odd under inversion symmetry. Note also that $\chi^{(3)}_{\alpha\beta\gamma\delta}$ for a given material has the same transformation properties the elastic constants of that material. The nonzero elements of $\chi^{(2)}_{\alpha\beta\gamma}$ and $\chi^{(3)}_{\alpha\beta\gamma\delta}$ for various crystal symmetries are compiled in Y.R. Shen, *The Principles of Nonlinear Optics* (Wiley, New York, 1984).

[27] See, for example, N. Bloembergen, *Nonlinear Optics* (The Advance Book Program), Addison-Wesley (1992), ISBN 0-201-57868-9.





We can analyze the response of the oscillator by expanding the displacement in powers of the electric field $\mathrm{E}(t)$ -- *viz.*

$$x(t) = x^{(1)}(t) + x^{(2)}(t) + x^{(3)}(t) + \cdots \qquad \text{[VII-6]}$$

where $x^{(n)}(t)$ is proportional to the nth power of the field $\mathrm{E}(t)$. Inserting this expression into Equation [VII-5] and equating like powers of $\mathrm{E}(t)$, we obtain the following hierarchy of equations:

$$\ddot{x}^{(1)}(t) + \Gamma\,\dot{x}^{(1)}(t) + \omega_o^2\,x^{(1)}(t) = (q/M)\,\mathrm{E}(t) \qquad \text{[VII-7a]}$$

$$\ddot{x}^{(2)}(t) + \Gamma\,\dot{x}^{(2)}(t) + \omega_o^2\,x^{(2)}(t) + a\left[x^{(1)}(t)\right]^2 = 0 \qquad \text{[VII-7b]}$$

$$\ddot{x}^{(3)}(t) + \Gamma\,\dot{x}^{(3)}(t) + \omega_o^2\,x^{(3)}(t) + 2a\,x^{(1)}(t)\,x^{(2)}(t) + b\left[x^{(1)}(t)\right]^3 = 0 \qquad \text{[VII-7c]}$$

$$\vdots$$

$$\vdots$$

In the frequency domain we see that

$$-\omega^2 x^{(1)}(\omega) - i\omega\,\Gamma\, x^{(1)}(\omega) + \omega_o^2\, x^{(1)}(\omega) = (q/M)\,\mathrm{E}\,(\omega) \qquad \text{[VII-8a]}$$

or

$$x^{(1)}(\omega) = (q/M)\,\mathrm{E}\,(\omega)\left[\omega_o^2 - \omega^2 - i\omega\,\Gamma\right]^{-1} = (q/M)\,\mathrm{E}\,(\omega)\,\mathcal{F}(\omega_0,\omega;\Gamma) \qquad \text{[VII-8b]}$$

where $\mathcal{F}(u,v;w) \equiv \left[u^2 - v^2 - i\,v\,w\right]^{-1}$.[28] Thus, knowing $x^{(1)}(\omega)$, we may then consider $x^{(1)}(t)$ a driving term in Equation [VII-7b] -- *viz*.

$$\ddot{x}^{(2)}(t) + \Gamma\,\dot{x}^{(2)}(t) + \omega_o^2\, x^{(2)}(t) = -a\left[x^{(1)}(t)\right]^2 \qquad \text{[VII-9]}$$

Therefore, in the frequency domain

$$x^{(2)}(\omega) = -a\,\mathcal{F}(\omega_0,\omega;\Gamma)\int d\omega'\, x^{(1)}(\omega')\, x^{(1)}(\omega-\omega') \qquad \text{[VII-10a]}$$

which, in view of Equation [11-8b], becomes

$$\begin{aligned} x^{(2)}(\omega) = -a\,(q/M)^2\,\mathcal{F}[\omega_0,\omega;\Gamma] & \\ \times \int d\omega'\,\mathcal{F}[\omega_0,\omega';\Gamma]\,\mathcal{F}[\omega_0,(\omega-\omega');\Gamma]\,\mathrm{E}\,(\omega')\,\mathrm{E}\,(\omega-\omega') & \end{aligned} \quad .\text{[VII-10b]}$$

We may treat the third order terms in a similar manner. We write Equation [VII-7c] as

$$\ddot{x}^{(3)}(t) + \Gamma\,\dot{x}^{(3)}(t) + \omega_o^2\, x^{(3)}(t) = -2a\, x^{(1)}(t)\, x^{(2)}(t) - b\left[x^{(1)}(t)\right]^3 \qquad \text{[VII-11]}$$

---

28 Note that near resonance

$$\mathcal{F}(\omega_0,\omega;\Gamma) \approx \frac{i}{2\omega}\left[i\,(\omega_0-\omega) + \Gamma/2\right]^{-1} = \frac{i}{2\omega}\left\{\frac{\Gamma/2 - i\,(\omega_0-\omega)}{(\omega_0-\omega)^2 + (\Gamma/2)^2}\right\} = \frac{i}{2\omega}\,\mathcal{D}(\omega_0-\omega;\Gamma/2)$$

where $\mathcal{D}$ is the so called complex Lorentzian.

In the frequency domain

$$x^{(3)}(\omega) = -2a\ \mathcal{F}(\omega_0,\omega;\Gamma) \int d\omega'\ x^{(1)}(\omega')\ x^{(2)}(\omega-\omega') \qquad \text{. VII-12]}$$
$$- b\ \mathcal{F}(\omega_0,\omega;\Gamma) \int\int d\omega'\ d\omega''\ x^{(1)}(\omega')\ x^{(1)}(\omega'')\ x^{(1)}(\omega-\omega'-\omega'')$$

Using Equations [VII-8b] and [VII-10b], we obtain

$$\begin{aligned} x^{(3)}(\omega) = 2a^2\ (q/M)^3\ \mathcal{F}[\omega_0,\omega;\Gamma] \int\int d\omega'\ d\omega''\ \mathcal{F}[\omega_0,(\omega-\omega');\Gamma]\ \mathcal{F}[\omega_0,\omega';\Gamma] \\ \times \mathcal{F}[\omega_0,\omega'';\Gamma]\ \mathcal{F}[\omega_0,(\omega'-\omega'');\Gamma]\ \mathrm{E}(\omega-\omega')\ \mathrm{E}(\omega'')\ \mathrm{E}(\omega'-\omega'') \\ - b\,(q/M)^3\ \mathcal{F}[\omega_0,\omega;\Gamma] \int\int d\omega'\ d\omega''\ \mathcal{F}[\omega_0,\omega';\Gamma]\ \mathcal{F}[\omega_0,\omega'';\Gamma] \\ \times \mathcal{F}[\omega_0,(\omega-\omega'-\omega'');\Gamma]\,\mathrm{E}(\omega')\ \mathrm{E}(\omega'')\ \mathrm{E}(\omega-\omega'-\omega'') \end{aligned} \qquad \text{[VII-13]}$$

In particular, for an input

$$\begin{aligned} \mathrm{E}(t) &= |\mathrm{E}_1|\cos(\omega_1 t+\phi_1) + |\mathrm{E}_2|\cos(\omega_2 t+\phi_2) \\ &= \frac{1}{2}\left[\mathrm{E}_1 \exp(-i\omega_1 t) + \mathrm{E}_1\ \exp(+i\omega_1 t)\right] + \frac{1}{2}\left[\mathrm{E}_2\ \exp(-i\omega_2 t) + \mathrm{E}_2\ \exp(+i\omega_2 t)\right] \end{aligned} \qquad \text{[VII-14]}$$

a. **Second harmonic generation** is due to the terms

$$x^{(2)}(2\omega_1) = -a\ (q/2M)^2\ \mathcal{F}[\omega_0,2\omega_1;\Gamma]\ \mathcal{F}^2[\omega_0,\omega_1;\Gamma]\ \mathrm{E}_1^2(\omega_1) \qquad \text{[VII-15a]}$$

and

$$x^{(2)}(2\omega_2) = -a\ (q/2M)^2\ \mathcal{F}[\omega_0,2\omega_2;\Gamma]\ \mathcal{F}^2[\omega_0,\omega_2;\Gamma]\ \mathrm{E}_2^2(\omega_2) \qquad \text{[VII-15b]}$$

b. **Sum frequency generation** is due to the term

$$x^{(2)}(\omega_1+\omega_2) = -a\,(q/2M)^2\; \mathcal{F}\left[\omega_0,(\omega_1+\omega_2);\Gamma\right] \times \mathcal{F}\left[\omega_0,\omega_1;\Gamma\right]\,\mathcal{F}\left[\omega_0,\omega_2;\Gamma\right]\,\mathrm{E}_1(\omega_1)\,\mathrm{E}_2(\omega_2) \qquad \text{[VII-16a]}$$

c. **Difference frequency generation** is due to the term

$$x^{(2)}(\omega_1-\omega_2) = -a\,(q/2M)^2\; \mathcal{F}\left[\omega_0,(\omega_1-\omega_2);\Gamma\right] \times \mathcal{F}\left[\omega_0,\omega_1;\Gamma\right]\,\mathcal{F}\left[\omega_0,\omega_2;\Gamma\right]\,\mathrm{E}_1(\omega_1)\,\mathrm{E}_2(\omega_2) \qquad \text{[VII-16b]}$$

d. **Optical rectification** or **dc generation** is due to the terms

$$x^{(2)}(0) = -a\,(q/2M)^2\; \mathcal{F}\left[\omega_0,0;\Gamma\right]\left|\mathcal{F}\left[\omega_0,\omega_1;\Gamma\right]\right|^2\left|\mathrm{E}_1(\omega_1)\right|^2 \qquad \text{[VII-17a]}$$

and

$$x^{(2)}(0) = -a\,(q/2M)^2\; \mathcal{F}\left[\omega_0,0;\Gamma\right]\left|\mathcal{F}\left[\omega_0,\omega_2;\Gamma\right]\right|^2\left|\mathrm{E}_2(\omega_2)\right|^2 \qquad \text{[VII-17b]}$$

e. **Third harmonic generation** is due to the terms

$$x^{(3)}(3\omega_1) = (q/M)^3\left\{2a^2\;\mathcal{F}\left[\omega_0,2\omega_1;\Gamma\right]-b\right\} \times \mathcal{F}\left[\omega_0,3\omega_1;\Gamma\right]\,\mathcal{F}^3\left[\omega_0,\omega_1;\Gamma\right]\mathrm{E}_1^3(\omega_1) \qquad \text{[VII-18a]}$$

and

$$x^{(3)}(3\omega_2) = (q/M)^3\left\{2a^2\;\mathcal{F}\left[\omega_0,2\omega_2;\Gamma\right]-b\right\} \times \mathcal{F}\left[\omega_0,3\omega_2;\Gamma\right]\,\mathcal{F}^3\left[\omega_0,\omega_2;\Gamma\right]\,\mathrm{E}_2^3(\omega_2) \qquad \text{[VII-18b]}$$

f. **Intensity dependent propagation** is due to the terms

$$x^{(3)}(\omega_1) = (q/2M)^3\left\{2a^2\;\mathcal{F}\left[\omega_0,2\omega_1;\Gamma\right]-b\right\} \times\left|\mathcal{F}\left[\omega_0,\omega_1;\Gamma\right]\right|^2\mathcal{F}^2\left[\omega_0,\omega_1;\Gamma\right]\left|\mathrm{E}_1(\omega_1)\right|^2\mathrm{E}_1(\omega_1) \qquad \text{[VII-19a]}$$

and

$$x^{(3)}(\omega_2) = (q/2M)^3 \left\{2a^2\, \mathcal{F}[\omega_0, 2\omega_2; \Gamma] - b\right\} \times \left|\mathcal{F}[\omega_0, \omega_2; \Gamma]\right|^2 \mathcal{F}^2[\omega_0, \omega_2; \Gamma]\, |\mathrm{E}_1(\omega_2)|^2\, \mathrm{E}_1(\omega_2) \qquad \text{[VII-19b]}$$

g. **Raman generation** (inelastic scattering) involves terms like

$$x^{(3)}(2\omega_1 - \omega_2) = 2\,(q/M)^3 \left\{2a^2\, \mathcal{F}[\omega_0, (\omega_1 - \omega_2); \Gamma] + a^2\, \mathcal{F}[\omega_0, 2\omega_1; \Gamma] - b\right\} \times \mathcal{F}[\omega_0, (2\omega_1 - \omega_2); \Gamma]\, \mathcal{F}^2[\omega_0, \omega_1; \Gamma]\, \mathcal{F}\,[\omega_0, \omega_2; \Gamma]\, \mathrm{E}_1^2(\omega_1)\, \mathrm{E}_2(\omega_2) \qquad \text{[VII-20]}$$

Note that, according to this simple anharmonic oscillator model, Raman generation may be enhanced by a resonance at a frequency $(\omega_1 - \omega_2)$! Also note that, for this model (see Equation [VII-13a] above), the ratio

$$\frac{\chi^{(2)}(\omega)}{\chi^{(1)}(\omega)\, \chi^{(1)}(\omega - \omega_1)\, \chi^{(1)}(\omega_1)} = -a\left(M/N^2 q^3\right) \qquad \text{[VII-21]}$$

is a constant independent of frequency! This observation is consistent with the famous empirical **Miller Rule** which declares that the ratio

$$\Delta_{ijk} = \frac{\chi^{(2)}_{iijk}(\omega_3 = \omega_1 + \omega_2)}{\chi^{(1)}_{ii}(\omega_3)\, \chi^{(1)}_{jj}(\omega_1)\, \chi^{(1)}_{kk}(\omega_2)} \qquad \text{[VII-22]}$$

has only a weak dispersion and is almost a constant for a wide range of materials!

SECOND HARMONIC GENERATION -- PERTURBATION ANALYSIS

We may write the **nonlinear**, macroscopic Maxwell equations in the form

$$\nabla^2 \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) - \vec{\nabla}\left[\vec{\nabla}\cdot \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right] + \left(\omega^2/c^2\varepsilon_0\right)\bar{\bar{\varepsilon}}\cdot \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = -\mu_0\,\omega^2\, \vec{\mathbf{P}}^{(\mathrm{NL})}(\vec{\mathbf{r}},\omega) \qquad \text{[VII-23]}$$

$$\vec{\nabla}\cdot\left[\bar{\bar{\varepsilon}}\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right] = -\vec{\nabla}\cdot\vec{\mathbf{P}}^{(\mathrm{NL})}(\vec{\mathbf{r}},\omega) \qquad \text{[VII-24]}$$

Suppose that we have an input driving or pump field

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega_1) = \hat{\mathbf{e}}^{(1)}\,\mathrm{E}(\omega_1)\,\exp\left(i\left(k_1 z - \omega_1 t\right)\right) + c.c. \qquad \text{[VII-25]}$$

Then the components of the polarization with frequency $2\omega_1$ are given by

$$\vec{\mathbf{P}}^{(\mathrm{NL})}(z,\omega_2 = 2\omega_1) = \mathrm{E}^2(\omega_1)\;\bar{\bar{\chi}}^{(2)}\!:\!\hat{\mathbf{e}}^{(1)}\,\hat{\mathbf{e}}^{(1)}\,\exp\left[i\left(2k_1 z - 2\omega_1 t\right)\right] + c.c. \qquad \text{[VII- 26]}$$

It is a convenience to resolve this $\mathbf{P}^{(\mathrm{NL})}$ and the resultant second harmonic field into longitudinal and transverse components -- *viz*.

$$\vec{\mathbf{P}}^{(\mathrm{NL})}(z,\omega_2) = \hat{\mathbf{z}}\,\mathrm{P}_{||}^{(\mathrm{NL})}(z,\omega_2) + \vec{\mathbf{P}}_{\perp}^{(\mathrm{NL})}(z,\omega_2)\ . \qquad \text{[VII-27a]}$$

and

$$\vec{\mathbf{E}}(z,\omega_2) = \hat{\mathbf{z}}\,\mathrm{E}_{||}(z,\omega_2) + \vec{\mathbf{E}}_{\perp}(z,\omega_2) \qquad \text{[VII-27b]}$$

Therefore, we may write the ω_2 or second harmonic components of Equations [VII-23] and [VII-24]

$$\frac{\partial^2}{\partial z^2}\vec{\mathbf{E}}_{\perp}(z,\omega_2) + \frac{\omega_2^2}{c^2}\frac{\varepsilon_{\perp}(\omega_2)}{\varepsilon_0}\vec{\mathbf{E}}_{\perp}(z,\omega_2) + \mu_0\,\omega_2^2\,\vec{\mathbf{P}}_{\perp}^{(\mathrm{NL})}(z,\omega_2)$$
$$+\left\{\frac{\omega_2^2}{c^2}\frac{\varepsilon_{||}(\omega_2)}{\varepsilon_0}\mathrm{E}_{||}(z,\omega_2) + \mu_0\,\omega_2^2\,\mathrm{P}_{||}^{(\mathrm{NL})}(z,\omega_2)\right\}\hat{\mathbf{z}} = 0 \qquad \text{[VII-28a]}$$

and

$$\frac{\partial}{\partial z}\left\{\varepsilon_{||}(\omega_2)\,\mathrm{E}_{||}(z,\omega_2) + \mathrm{P}_{||}^{(\mathrm{NL})}(z,\omega_2)\right\} = 0 \qquad \text{[VII-28b]}$$

To satisfy these equations two conditions must hold -- *viz*.

$$\mathrm{E}_{\|}(z,\omega_2) = -\frac{1}{\varepsilon_{\|}(\omega_2)}\,\mathrm{P}_{\|}^{(\mathrm{NL})}(z,\omega_2) \qquad \text{[VII-29a]}$$

and

$$\left[\frac{\partial^2}{\partial z^2} + k_2^2\right]\vec{\mathbf{E}}\,(z,\omega_2) = -\mu_0\,\omega_2^2\,\vec{\mathbf{P}}^{(\mathrm{NL})}(z,\omega_2) \qquad \text{[VII-29b]}$$

where

$$k_2 \equiv \frac{\omega_2}{c}\sqrt{\varepsilon\,(\omega_2)/\varepsilon_0} = 2\omega_1\sqrt{\mu_0\,\varepsilon\,(2\omega_1)} \qquad \text{[VII-30]}$$

We now write $\vec{\mathbf{E}}\,(z,\omega_2) = \hat{\mathbf{e}}^{(2)}\,\mathcal{E}(z,\omega_2)\exp(i\,k_2 z)$ where $\mathcal{E}(z,\omega_2)$ is a slowly varying function of z -- *viz*.

$$\begin{aligned}\frac{\partial^2}{\partial z^2}\vec{\mathbf{E}}\,(z,\omega_2) &= -\hat{\mathbf{e}}^{(2)}\left\{k_2^2\,\mathcal{E}(z,\omega_2) - 2\,i\,k_2\frac{\partial}{\partial z}\mathcal{E}(z,\omega_2) - \frac{\partial^2}{\partial z^2}\mathcal{E}(z,\omega_2)\right\}\exp(i\,k_2 z)\\ &\approx -\left\{k_2^2\,\mathcal{E}(z,\omega_2) - 2\,i\,k_2\frac{\partial}{\partial z}\mathcal{E}(z,\omega_2)\right\}\exp(i\,k_2 z)\end{aligned} \qquad \text{[VII-31]}$$

so that Equation [VII-29b] may be written

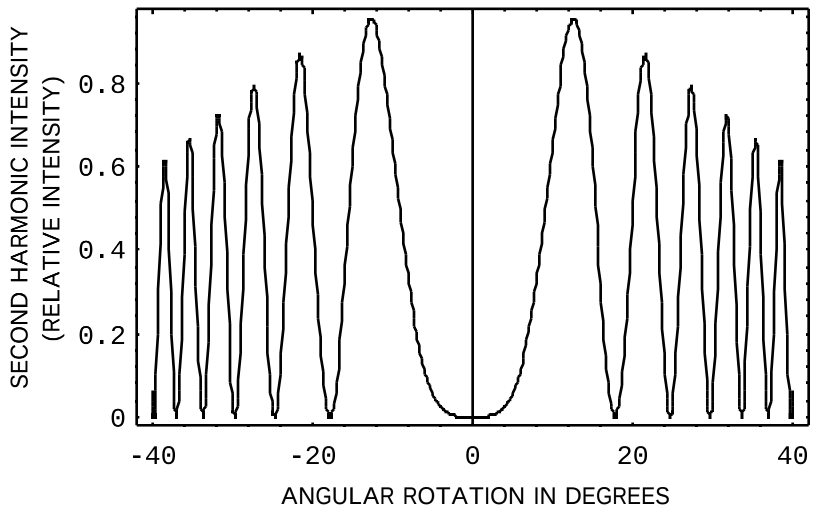
$$\begin{aligned}\frac{\partial}{\partial z}\mathcal{E}(z,\omega_2) &= \frac{i}{2k_2}\mu_0\,\omega_2^2\,\hat{\mathbf{e}}^{(2)}\!:\vec{\mathbf{P}}^{(\mathrm{NL})}(z,\omega_2)\exp(-i\,k_2 z)\\ &= \frac{i\,\mu_0\,\omega_2^2}{2k_2}\,\mathrm{E}^2(\omega_1)\,\hat{\mathbf{e}}^{(2)}\!:\overset{\equiv}{\chi}{}^{(2)}\!:\hat{\mathbf{e}}^{(1)}\,\hat{\mathbf{e}}^{(1)}\exp(i\,\Delta k z)\end{aligned} \qquad \text{[VII-32]}$$

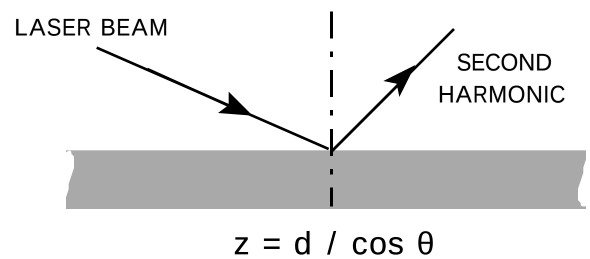
where $\Delta k \equiv 2k_1 - k_2 = \dfrac{2\omega_1}{c}\left[n_{\mathrm{FH}}(\omega_1) - n_{\mathrm{SH}}(2\omega_1)\right]$.

If we assume that the driving field stays constant, we can directly integrate Equation [VII-32] to obtain the **exceedingly famous and important equation** for the spatial variation of the second harmonic field -- *viz*.

$$\mathcal{E}(z,\omega_2) = z\left\{\frac{i\,\mu_0\,\omega_2^2}{4\,k_2}\,\hat{\mathbf{e}}^{(2)}:\overline{\overline{\chi}}^{(2)}:\hat{\mathbf{e}}^{(1)}\,\hat{\mathbf{e}}^{(1)}\,\exp\left(i\,\Delta k\,z/2\right)\right\}\mathrm{E}^2(\omega_1)\,\mathrm{sinc}\left[\Delta k z/2\right] \quad .\text{[VII-33]}$$

MAKER[29] FRINGES

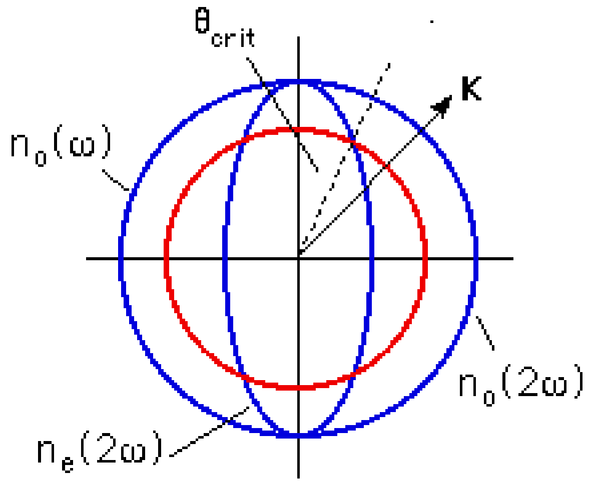


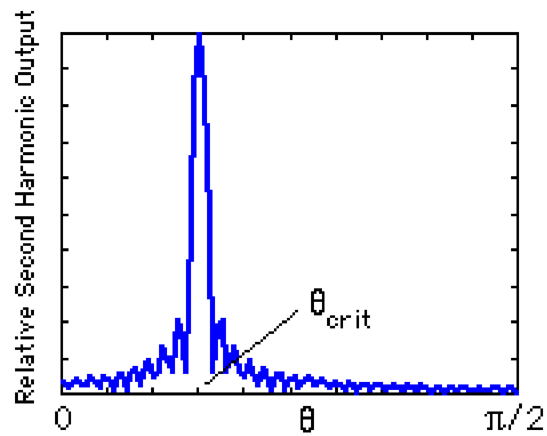


SECOND HARMONIC GENERATION - COUPLED WAVE ANALYSIS

When the process of second harmonic generation takes place under conditions of ***perfect phase match*** -- *i.e.* $k_2 = 2k_1$ -- the perturbation result breaks down if the path is sufficiently long. Under these circumstances the pump beam will be depleted as the second harmonic grows and a solution of Equation [VII-32] must take into account the spatial variation of $\mathrm{E}^2(\omega_1)$. To that end we assume perfect phase matching

[29] P.D. Maker, R.W. Terhune, N. Nisenoff, and C.M. Savage, *Phys. Rev. Lett.*, **8**, 21 (1965).





and rewrite Equation [VII-32] as

$$\frac{\partial}{\partial z}\mathcal{E}(z,\omega_2) = \frac{i\,\mu_0\,\omega_1^2}{k_1}\,\hat{\mathbf{e}}^{(2)}:\overline{\overline{\chi}}^{(2)}:\hat{\mathbf{e}}^{(1)}\,\hat{\mathbf{e}}^{(1)}\,E^2(z,\omega_1) \qquad \text{[VII-34]}$$

Of course, as the second harmonic grows Equation [VII-3] tells us that a nonlinear polarization at the pump frequency is generated -- *viz*.

$$\vec{\mathbf{P}}^{(NL)}(z,\omega_1) = E(z,2\omega_1)\,E\,(z,\omega_1)\left[\overline{\overline{\chi}}^{(2)}:\hat{\mathbf{e}}^{(2)}\,\hat{\mathbf{e}}^{(1)} + \overline{\overline{\chi}}^{(2)}:\hat{\mathbf{e}}^{(1)}\,\hat{\mathbf{e}}^{(2)}\right] \qquad \text{[VII-35]}$$

Repeating the arguments associated with Equations [VII-28] through [VII-31] we may write

$$\frac{\partial}{\partial z}\vec{\mathbf{E}}(z,\omega_1) = \frac{i}{2k_1}\,\mu_0\,\omega_1^2\,\vec{\mathbf{P}}^{\,(NL)}(z,\omega_1) \qquad \text{[VII-36]}$$

Combining these two equations, we find an equation governing the spatial evolution of $\vec{\mathbf{E}}(z,\omega_1)$ -- *viz.*

$$\frac{\partial}{\partial z}\,\mathrm{E}(z,\omega_1) = \frac{i\,\mu_0\,\omega_1^2}{2k_1}\left[\hat{\mathbf{e}}^{(1)}:\bar{\bar{\bar{\chi}}}^{(2)}:\hat{\mathbf{e}}^{(2)}\;\hat{\mathbf{e}}^{(1)} + \hat{\mathbf{e}}^{(1)}:\bar{\bar{\bar{\chi}}}^{(2)}:\hat{\mathbf{e}}^{(1)}\;\hat{\mathbf{e}}^{(2)}\right]\mathrm{E}(z,2\omega_1)\,\mathrm{E}^{*}(z,\omega_1)\;. \qquad \text{[VII-37]}$$

Equations [VII-34] and [VII-37] are then the coupled differential equation which describe the coupling of the first and second harmonic fields. Handling all the "vectorness" in these two equation would obscure important issues. Thus, we consider a pair of somewhat less complicated equations which incorporate the essence of the problem -- *viz.*

$$\frac{\partial}{\partial z}\,\mathrm{E}(z,\omega_2) = \frac{i\,\mu_0\,\omega_1^2}{k_1}\,\chi^{(2)}\,\mathrm{E}^2(z,\omega_1) \qquad \text{[VII-38a]}$$

$$\frac{\partial}{\partial z}\,\mathrm{E}(z,\omega_1) = \frac{i\,\mu_0\,\omega_1^2}{k_1}\,\chi^{(2)}\,\mathrm{E}(z,2\omega_1)\,\mathrm{E}^{*}(z,\omega_1) \qquad \text{[VII-38b]}$$

To solve these equation we first write $\mathrm{E}(z,\omega_1) = \mathrm{E}(\omega_1)\,\mathrm{f}_1(z)$ and $\mathrm{E}(z,\omega_2) = \mathrm{E}(\omega_1)\,\mathrm{f}_2(z)$ with the boundary conditions $\mathrm{f}_1(0) = 1$ and $\mathrm{f}_2(0) = 0$. Thus, the coupled equations reduce to the dimensionless form

$$\frac{d}{d\xi}\,\mathrm{f}_2(\xi) = i\,\mathrm{f}_1^2(\xi) \qquad \text{[VII-39a]}$$

and

$$\frac{d}{d\xi}\,\mathrm{f}_1(\xi) = i\,\mathrm{f}_2(\xi)\,\mathrm{f}_1^{*}(\xi) \qquad \text{[VII-39b]}$$

where $\xi = z/\mathrm{L}_c$ and $\mathrm{L}_c^{-1} = \left[\mu_0\,\omega_1^2\chi^{(2)}\,\mathrm{E}(\omega_1)\right]\big/k_1$. We next separate $\mathrm{f}_1(\xi)$ and $\mathrm{f}_2(\xi)$ into phase and amplitude parts as

$$f_{1,2}(\xi) = u_{1,2}(\xi)\, \exp\left[i\, \varphi_{1,2}(\xi)\right] \qquad \text{[VII-40]}$$

and substitute into Equations [VII-39a] and [VII-39b] -- *viz*.

$$\frac{d}{d\xi} u_2(\xi) + i\, u_2(\xi)\, \frac{d}{d\xi}\varphi_2(\xi) = i\, u_1^2(\xi)\, \exp\left\{ i\left[2\varphi_1(\xi) - \varphi_2(\xi)\right]\right\} \qquad \text{[VII-41a]}$$

$$\frac{d}{d\xi} u_1(\xi) + i\, u_1(\xi)\, \frac{d}{d\xi}\varphi_1(\xi) = i\, u_1(\xi)\, u_2(\xi)\, \exp\left\{ -i\left[2\varphi_1(\xi) - \varphi_2(\xi)\right]\right\} \qquad \text{[VII-41b]}$$

Equating real and imaginary parts of these equations, we find

$$\frac{d}{d\xi} u_2(\xi) = -u_1^2(\xi) \sin\left[2\varphi_1(\xi) - \varphi_2(\xi)\right] \qquad \text{[VII-42a]}$$

$$u_2(\xi)\, \frac{d}{d\xi}\varphi_2(\xi) = u_1^2(\xi) \cos\left[2\varphi_1(\xi) - \varphi_2(\xi)\right] \qquad \text{[VII-42b]}$$

$$\frac{d}{d\xi} u_1(\xi) = u_1(\xi)\, u_2(\xi)\, \sin\left[2\varphi_1(\xi) - \varphi_2(\xi)\right] \qquad \text{[VII-42c]}$$

$$\frac{d}{d\xi}\varphi_1(\xi) = u_2(\xi)\, \cos\left[2\varphi_1(\xi) - \varphi_2(\xi)\right] \qquad \text{[VII-42d]}$$

Since the phase enters only in the combination $\psi(\xi) = 2\varphi_1(\xi) - \varphi_2(\xi)$ these four equations reduce to three -- *viz*.

$$\frac{d}{d\xi} u_2(\xi) = -u_1^2(\xi)\, \sin\psi(\xi) \qquad \text{[VII-43a]}$$

$$\frac{d}{d\xi} u_1(\xi) = u_1(\xi)\, u_2(\xi)\, \sin\psi(\xi) \qquad \text{[VII-43b]}$$

$$\frac{d}{d\xi}\psi(\xi) = \left[2u_2(\xi) - \frac{u_1^2(\xi)}{u_2(\xi)} \right] \cos\psi(\xi) \qquad \text{[VII-43c]}$$

Combining these three equations, we obtain

$$\tan\psi(\xi)\,\frac{d}{d\xi}\psi(\xi) - 2\frac{1}{u_1(\xi)}\,\frac{d}{d\xi}u_1(\xi) - \frac{1}{u_2(\xi)}\,\frac{d}{d\xi}u_2(\xi) = 0 \qquad \text{[VII-44a]}$$

which is, obviously, equivalent to

$$\frac{d}{d\xi}\ln\left[\, u_2(\xi)\, u_1^2(\xi)\, \cos\psi(\xi) \,\right] = 0 \qquad \text{[VII-44b]}$$

or

$$u_2(\xi)\, u_1^2(\xi)\, \cos\psi(\xi) = \text{const.} \qquad \text{[VII-44c]}$$

Since $u_2(\xi)$ 0 as ξ 0 the "const' must be zero and, hence, $\psi(\xi)$ must be $\pi/2$ for all ξ. Thus, the original four coupled equations now reduce to two -- *viz*.

$$\frac{d}{d\xi}u_2(\xi) = u_1^2(\xi) \qquad \text{[VII-45a]}$$

and

$$\frac{d}{d\xi}u_1(\xi) = -u_1(\xi)\, u_2(\xi) \qquad \text{[VII-45b]}$$

Combining these equations, we obtain

$$u_1(\xi)\frac{d}{d\xi}u_1(\xi) + u_2(\xi)\frac{d}{d\xi}u_2(\xi) = \frac{d}{d\xi}\left[u_1^2(\xi) + u_2^2(\xi)\right] = 0 \qquad \text{[VII-46a]}$$

or

$$u_1^2(\xi) + u_2^2(\xi) = 1 \qquad \text{[VII-46b]}$$

which is an assertion of the principle of **energy conservation**. Taking this equation together with Equation [VII-45a] we see that

$$\frac{d\,u_2(\xi)}{1 - u_2^2(\xi)} = d\,\xi \qquad \text{[VII-47a]}$$

or

$$u_2(\xi) = \tanh \xi \qquad \text{[VII-47b]}$$

and

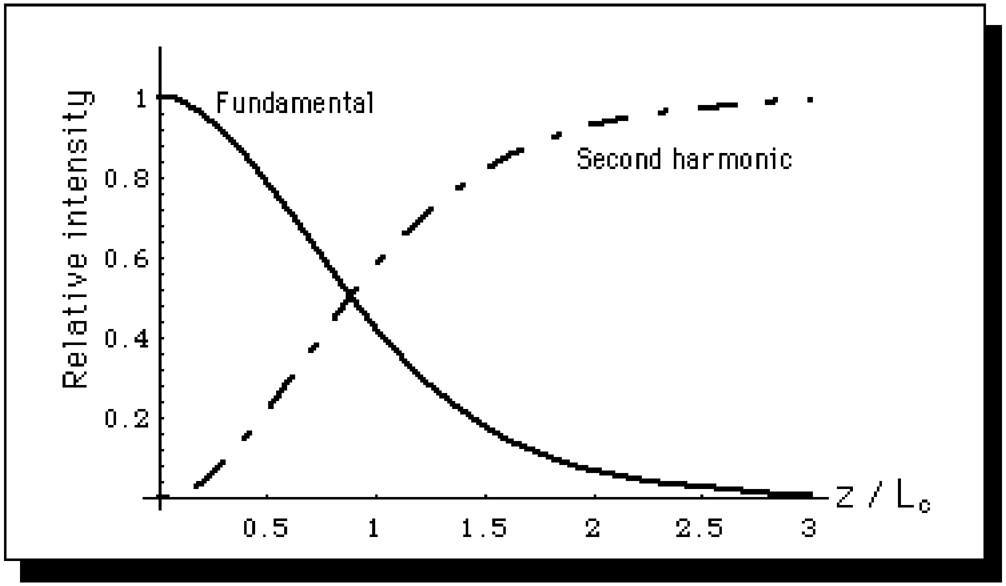
$$d \ln u_1(\xi) = -\tanh(\xi)\; d\,\xi \qquad u_1(\xi) = \frac{1}{\cosh(\xi)} \qquad \text{[VII-47c]}$$

Finally, returning to the original variables, we see that

$$\mathrm{E}(z, \omega_2) = i\,\mathrm{E}(\omega_1)\,\tanh(z/\mathrm{L}_c) \qquad \text{[VII-48a]}$$

and

$$\mathrm{E}(z, \omega_1) = \frac{\mathrm{E}(\omega_1)}{\cosh(z/\mathrm{L}_c)} \qquad \text{[VII-48b]}$$



where $L_c = \dfrac{k_1}{\mu_0\,\omega_1^2\,\chi^{(2)}\,E(\omega_1)}$

VIII. GUIDED WAVES IN PLANAR STRUCTURES

CHARACTERISTICS OF PLANE WAVE SOLUTIONS:

For the record, let us restate the frequency domain, macroscopic Maxwell's equations which are valid in the high frequency or *optical regime* for a linear, local, isotropic medium -- *viz*.

$$\vec{\nabla}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=\vec{\nabla}\times\varepsilon^{-1}(\vec{\mathbf{r}},\omega)\,\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega)=-i\,\omega\,\vec{\mathbf{B}}(\vec{\mathbf{r}},\omega)=-i\,\omega\mu_0\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)\qquad\text{[VIII- 1a]}$$

$$\begin{aligned}\vec{\nabla}\times\vec{\mathbf{B}}(\vec{\mathbf{r}},\omega)&=\mu_0\,\vec{\nabla}\times\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=\mu_0\,\vec{\mathbf{J}}(\vec{\mathbf{r}},\omega)+i\,\omega\mu_0\varepsilon(\vec{\mathbf{r}},\omega)\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\\&=\mu_0\,\vec{\mathbf{J}}(\vec{\mathbf{r}},\omega)+i\omega\mu_0\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega)\end{aligned}\qquad\text{[VIII- 1b]}$$

$$\vec{\nabla}\cdot\varepsilon(\vec{\mathbf{r}},\omega)\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=\vec{\nabla}\cdot\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega)=\rho(\vec{\mathbf{r}},\omega)\qquad\text{[VIII- 1c]}$$

$$\vec{\nabla}\cdot\vec{\mathbf{B}}(\vec{\mathbf{r}},\omega)=\mu_0\,\vec{\nabla}\cdot\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=0\qquad\text{[VIII- 1d]}$$

Further, in regions free of **explicit sources** of current and charge we may write

$$\vec{\nabla}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=-i\,\omega\mu_0\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)\qquad\text{[VIII- 2a]}$$

$$\vec{\nabla}\times\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=i\,\omega\varepsilon_{\text{eff}}(\vec{\mathbf{r}},\omega)\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\qquad\text{[VIII- 2b]}$$

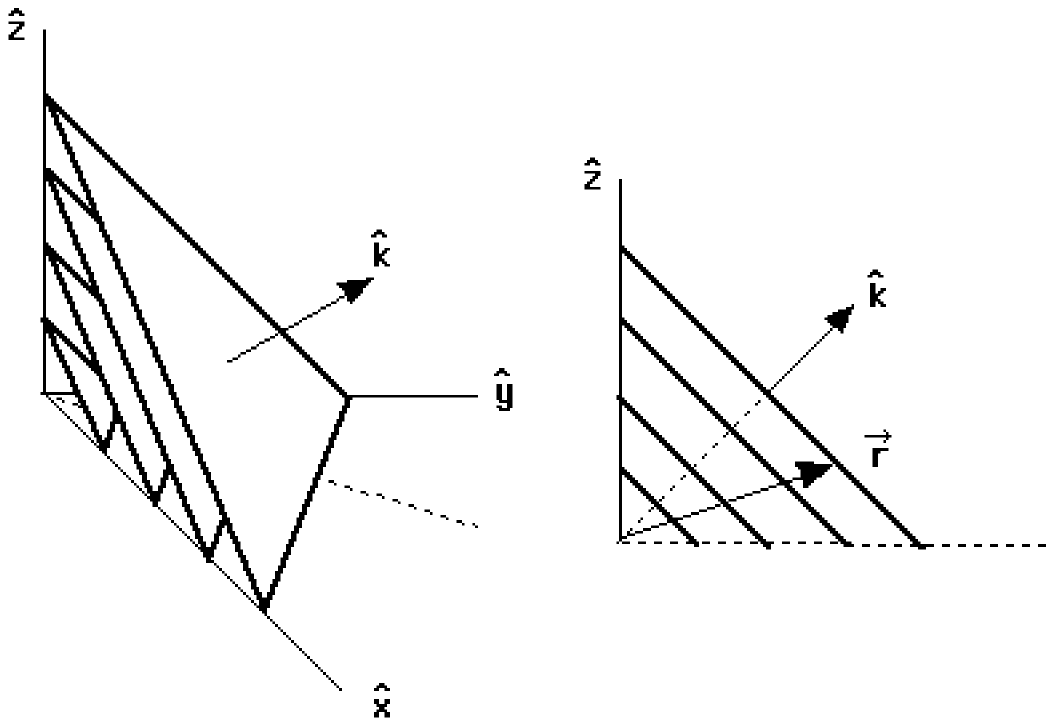
$$\vec{\nabla}\cdot\varepsilon(\vec{\mathbf{r}},\omega)\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=0\qquad\text{[VIII- 2c]}$$

$$\vec{\nabla}\cdot\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=0\qquad\text{[VIII- 2d]}$$

where $\varepsilon_{\text{eff}}(\vec{\mathbf{r}},\omega)\equiv\varepsilon_{\text{eff}}(\vec{\mathbf{r}},\omega)-i\,\sigma(\vec{\mathbf{r}},\omega)/\omega$. In this set of lectures it is our intention to **explore in some depth** plane wave propagation within a uniform medium -- *i.e.* $\varepsilon_{\text{eff}}(\vec{\mathbf{r}},\omega)\equiv\varepsilon_{\text{eff}}(\omega)$. To that end we consider a plane wave solution in the form

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\mathbf{E}}(\omega)\,\exp\left(-i\,\vec{\mathbf{r}}\;\;\vec{\mathbf{k}}\right) = \vec{\mathbf{E}}(\omega)\,\exp\left[-i\left(x\,k_x + y\,k_y + z\,k_z\right)\right] \qquad \text{[VIII- 3]}$$

which may pictorially represented as



Therefore

$$\vec{\nabla}\;\;\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\nabla}\;\;\left[\vec{\mathbf{E}}(\omega)\,\exp\left(-i\,\vec{\mathbf{r}}\;\;\vec{\mathbf{k}}\right)\right] = -i\,\vec{\mathbf{k}}\;\;\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[VIII- 4a]}$$

$$\vec{\nabla}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\nabla}\times\left[\vec{\mathbf{E}}(\omega)\,\exp\left(-i\,\vec{\mathbf{r}}\;\;\vec{\mathbf{k}}\right)\right] = -i\,\vec{\mathbf{k}}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[VIII- 4b]}$$

and the Maxwell's equations formulated in Equation [VIII-2] become

$$-i\,\vec{\mathbf{k}}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=-i\,\omega\mu_0\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) \qquad \text{[VIII- 5a]}$$

$$-i\,\vec{\mathbf{k}}\times\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=i\,\omega\varepsilon(\omega)\;\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[VIII- 5b]}$$

$$-i\,\vec{\mathbf{k}}\;\;\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=0 \qquad \text{[VIII- 5c]}$$

$$-i\,\vec{\mathbf{k}}\;\;\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=0 \qquad \text{[VIII- 5d]}$$

Operate through on both sides of Equation [VIII- 5a] with the operator "$\vec{\mathbf{k}}\times$" we obtain

$$\vec{\mathbf{k}}\times\left[\vec{\mathbf{k}}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right]=\omega\mu_0\;\vec{\mathbf{k}}\times\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) \qquad \text{[VIII- 6a]}$$

Using the "bac-cab" rule[30] and Equation [VIII- 5b] this becomes

$$\vec{\mathbf{k}}\left[\vec{\mathbf{k}}\;\;\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right]-\left[\vec{\mathbf{k}}\;\;\vec{\mathbf{k}}\right]\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=-\omega^2\mu_0\,\varepsilon_{\text{eff}}(\omega)\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[VIII- 6b]}$$

or finally

$$\left[\vec{\mathbf{k}}\;\;\vec{\mathbf{k}}\right]\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=\omega^2\mu_0\,\varepsilon_{\text{eff}}(\omega)\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad k^2=\omega^2\mu_0\,\varepsilon_{\text{eff}}(\omega) \qquad \text{[VIII- 6c]}$$

Substituting these results into Equation [VIII- 5a] we obtain

$$\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=(\omega\mu_0)^{-1}\,k\,\left[\hat{\mathbf{k}}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right]=\sqrt{\varepsilon_{\text{eff}}(\omega)/\mu_0}\;\left[\hat{\mathbf{k}}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right] \qquad \text{[VIII- 7]}$$

so that the **wave impedance** is given by

$$\eta(\omega)=\left|\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right|/\left|\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)\right|=\sqrt{\mu_0/\varepsilon_{\text{eff}}(\omega)}\;. \qquad \text{[VIII- 11]}$$

---

30 That is $\vec{\mathbf{a}}\times\left(\vec{\mathbf{b}}\times\vec{\mathbf{c}}\right)=\vec{\mathbf{b}}\left(\vec{\mathbf{a}}\;\;\vec{\mathbf{c}}\right)-\vec{\mathbf{c}}\left(\vec{\mathbf{a}}\;\;\vec{\mathbf{b}}\right)$.

Thus, the complete expression for an electromagnetic plane wave propagating in a direction $\hat{\mathbf{k}}$ in a uniform medium is given by

$$\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)=\vec{\mathbf{E}}(\omega)\,\exp\left[-j\left(\vec{\mathbf{r}}\;\;\vec{\mathbf{k}}-\omega\,t\right)\right] \qquad \text{[VIII- 9a]}$$

$$\vec{\mathbf{H}}\left(\vec{\mathbf{r}},t\right)=\left[\eta(\omega)\right]^{-1}\left[\hat{\mathbf{k}}\times\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)\right] \qquad \text{[VIII- 9b]}$$

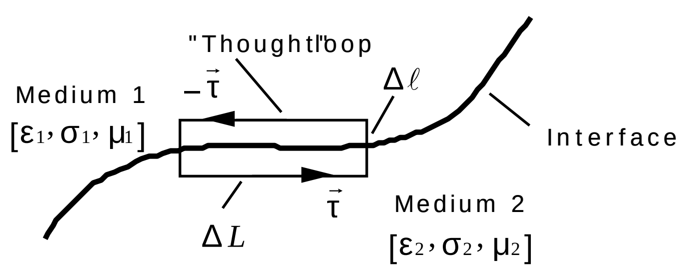
<u>ELECTROMAGNETIC INTERFACIAL CONTINUITY CONDITIONS:</u>

The previous section gives a **complete** plane wave solution within a **particular** uniform, linear, isotropic medium. The key remaining problem is to find how that solution may be extended into a second uniform, linear, isotropic medium. The conditions for extending the solution across an interface between two materials are give by consideration of the appropriate integral forms of Maxwell's equations -- *viz*.

$$\oint\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)\;\;d\vec{l}\;=-\frac{\partial}{\partial t}\iint\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)\;\;d\vec{A} \qquad \text{[VIII- 10a]}$$

$$\oint\vec{\mathbf{H}}\left(\vec{\mathbf{r}},t\right)\;\;d\vec{l}\;=\iint\vec{\mathbf{J}}\left(\vec{\mathbf{r}},t\right)\;\;d\vec{A}+\frac{\partial}{\partial t}\iint\vec{\mathbf{D}}\left(\vec{\mathbf{r}},t\right)\;\;d\vec{A} \qquad \text{[VIII- 10b]}$$

Applying these equations to the small ***thought* loop** that spans the interfacial surface, as illustrated below



it is seen that Equation [VIII- 10a] yields

$$\oint \vec{\mathbf{E}}(\vec{\mathbf{r}}, t) \quad d\vec{l} \quad \left\{\vec{\mathbf{E}}_2(\vec{\mathbf{r}}, t) - \vec{\mathbf{E}}_1(\vec{\mathbf{r}}, t)\right\} \quad \vec{\tau}\ \Delta L \quad 0 \qquad \text{[VIII- 11]}$$

unless $\vec{\mathbf{B}}(\vec{\mathbf{r}}, t)$ is **pathologically** large over the loop. Similarly, it is seen that Equation [VIII- 10b] yields

$$\oint \vec{\mathbf{H}}(\vec{\mathbf{r}}, t) \quad d\vec{l} \quad \left\{\vec{\mathbf{H}}_2(\vec{\mathbf{r}}, t) - \vec{\mathbf{H}}_1(\vec{\mathbf{r}}, t)\right\} \quad \vec{\tau}\ \Delta L \quad 0 \qquad \text{[VIII- 12]}$$

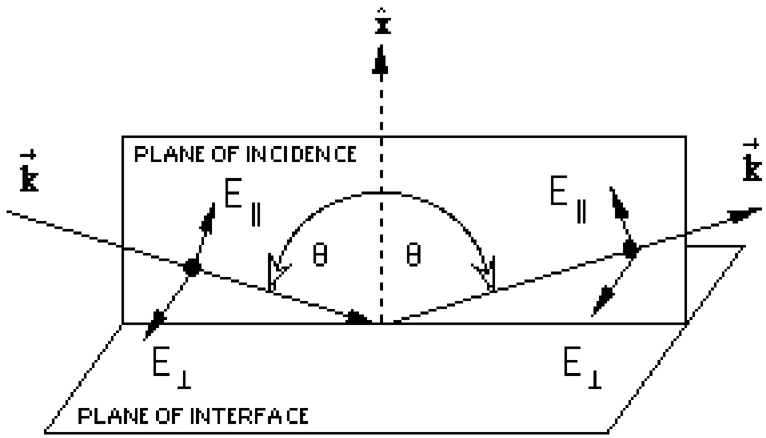
unless $\vec{\mathbf{J}}(\vec{\mathbf{r}}, t)$ and/or $\vec{\mathbf{D}}(\vec{\mathbf{r}}, t)$ are **pathologically** large over the loop.

> In words and in general, **the tangential component of the electric field strength $\vec{\mathbf{E}}(\vec{\mathbf{r}}, t)$ and the magnetic field strength $\vec{\mathbf{H}}(\vec{\mathbf{r}}, t)$ are continuous across an interfacial surface between two materials unless the electric current density $\vec{\mathbf{J}}(\vec{\mathbf{r}}, t)$, the magnetic flux density $\vec{\mathbf{B}}(\vec{\mathbf{r}}, t)$, or the electric flux density $\vec{\mathbf{D}}(\vec{\mathbf{r}}, t)$ are pathologically large near that interfacial surface.**

THE FRESNEL EQUATIONS:

Consider then a plane wave incident on a planar interfacial surface.

The Spatial Configuration:[31]



The Mathematical Representation of Fields:

In abstract vector form, the incident field is given by[32]

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{inc}} &= \left\{\vec{\mathbf{E}}^{\,\mathrm{inc}} - \eta_1\,\hat{\mathbf{k}}^{\mathrm{inc}} \times \vec{\mathbf{H}}_{||}^{\mathrm{inc}}\right\} \exp\left(-i\,k_1\,\hat{\mathbf{k}}^{\mathrm{inc}}\;\vec{\mathbf{r}}\right)\\ \vec{\mathbf{H}}^{\mathrm{inc}} &= \left\{\vec{\mathbf{H}}_{||}^{\mathrm{inc}} + \eta_1^{-1}\,\hat{\mathbf{k}}^{\mathrm{inc}} \times \vec{\mathbf{E}}^{\,\mathrm{inc}}\right\} \exp\left(-i\,k_1\,\hat{\mathbf{k}}^{\mathrm{inc}}\;\vec{\mathbf{r}}\right)\end{aligned} \qquad \text{[VIII- 13a]}$$

31 Note: In this figure we have taken the *plane of reflection* to be identical to the *plane of incidence*. While assumed here for simplicity, this important identity is establish in the analysis below.

32 A note on notation: The subscripts and $||$ refer to the polariztion of the electric field taken with respect to the *plane of incidence*. The field components are also called *transverse electric* or TE components and the $||$ field components are called *transverse magnetic* or TM components.

the reflected field is given by

$$\begin{aligned}\vec{\mathbf{E}}^{\text{ref}} &= \left\{\vec{\mathbf{E}}^{\text{ref}} - \eta_1\,\hat{\mathbf{k}}^{\text{ref}} \times \vec{\mathbf{H}}^{\text{ref}}_{||}\right\} \exp\left(-i\,k_1\,\hat{\mathbf{k}}^{\text{ref}}\,\vec{\mathbf{r}}\right)\\ \vec{\mathbf{H}}^{\text{ref}} &= \left\{\vec{\mathbf{H}}^{\text{ref}}_{||} + \eta_1^{-1}\,\hat{\mathbf{k}}^{\text{ref}} \times \vec{\mathbf{E}}^{\text{ref}}\right\} \exp\left(-i\,k_1\,\hat{\mathbf{k}}^{\text{ref}}\,\vec{\mathbf{r}}\right)\end{aligned} \qquad \text{[VIII- 14a]}$$

and the transmitted field is given by

$$\begin{aligned}\vec{\mathbf{E}}^{\text{tran}} &= \left\{\vec{\mathbf{E}}^{\text{tran}} - \eta_2\,\hat{\mathbf{k}}^{\text{tran}} \times \vec{\mathbf{H}}^{\text{tran}}_{||}\right\} \exp\left(-i\,k_2\,\hat{\mathbf{k}}^{\text{tran}}\,\vec{\mathbf{r}}\right)\\ \vec{\mathbf{H}}^{\text{tran}} &= \left\{\vec{\mathbf{H}}^{\text{tran}}_{||} + \eta_2^{-1}\,\hat{\mathbf{k}}^{\text{tran}} \times \vec{\mathbf{E}}^{\text{tran}}\right\} \exp\left(-i\,k_2\,\hat{\mathbf{k}}^{\text{tran}}\,\vec{\mathbf{r}}\right)\end{aligned} \qquad \text{[VIII- 15a]}$$

In coordinate form these equations become:

$$\begin{aligned}\vec{\mathbf{E}}^{\text{inc}} &= \left\{\mathrm{E}^{\text{inc}}\hat{\mathbf{y}} - \eta_1\left[-\cos\theta_{\text{inc}}\,\hat{\mathbf{x}} + \sin\theta_{\text{inc}}\,\hat{\mathbf{z}}\right]\times\left[\mathrm{H}^{\text{inc}}_{||}\hat{\mathbf{y}}\right]\right\} \exp\left[-i\,k_1\left(-x\cos\theta_{\text{inc}} + z\sin\theta_{\text{inc}}\right)\right]\\ \vec{\mathbf{H}}^{\text{inc}} &= \left\{\mathrm{H}^{\text{inc}}_{||}\hat{\mathbf{y}} + \eta_1^{-1}\left[-\cos\theta_{\text{inc}}\,\hat{\mathbf{x}} + \sin\theta_{\text{inc}}\,\hat{\mathbf{z}}\right]\times\left[\mathrm{E}^{\text{inc}}\hat{\mathbf{y}}\right]\right\} \exp\left[-i\,k_1\left(-x\cos\theta_{\text{inc}} + z\sin\theta_{\text{inc}}\right)\right]\end{aligned} \qquad \text{[VIII- 13b]}$$

$$\begin{aligned}\vec{\mathbf{E}}^{\text{ref}} &= \left\{\mathrm{E}^{\text{ref}}\hat{\mathbf{y}} - \eta_1\left[\cos\theta_{\text{ref}}\,\hat{\mathbf{x}} + \sin\theta_{\text{ref}}\,\hat{\mathbf{z}}\right]\times\left[\mathrm{H}^{\text{ref}}_{||}\hat{\mathbf{y}}\right]\right\} \exp\left[-i\,k_1\left(x\cos\theta_{\text{ref}} + z\sin\theta_{\text{ref}}\right)\right]\\ \vec{\mathbf{H}}^{\text{ref}} &= \left\{\mathrm{H}^{\text{ref}}_{||}\hat{\mathbf{y}} + \eta_1^{-1}\left[\cos\theta_{\text{ref}}\,\hat{\mathbf{x}} + \sin\theta_{\text{ref}}\,\hat{\mathbf{z}}\right]\times\left[\mathrm{E}^{\text{ref}}\hat{\mathbf{y}}\right]\right\} \exp\left[-i\,k_1\left(x\cos\theta_{\text{ref}} + z\sin\theta_{\text{ref}}\right)\right]\end{aligned} \qquad \text{[VIII- 14b]}$$

$$\begin{aligned}\vec{\mathbf{E}}^{\text{tran}} &= \left\{\mathrm{E}^{\text{tran}}\hat{\mathbf{y}} - \eta_2\left[-\cos\theta_{\text{tran}}\,\hat{\mathbf{x}} + \sin\theta_{\text{tran}}\,\hat{\mathbf{z}}\right]\times\left[\mathrm{H}^{\text{tran}}_{||}\hat{\mathbf{y}}\right]\right\} \exp\left[-i\,k_2\left(-x\cos\theta_{\text{tran}} + z\sin\theta_{\text{tran}}\right)\right]\\ \vec{\mathbf{H}}^{\text{tran}} &= \left\{\mathrm{H}^{\text{tran}}_{||}\hat{\mathbf{y}} + \eta_2^{-1}\left[-\cos\theta_{\text{tran}}\,\hat{\mathbf{x}} + \sin\theta_{\text{tran}}\,\hat{\mathbf{z}}\right]\times\left[\mathrm{E}^{\text{tran}}\hat{\mathbf{y}}\right]\right\} \exp\left[-i\,k_2\left(-x\cos\theta_{\text{tran}} + z\sin\theta_{\text{tran}}\right)\right]\end{aligned} \qquad \text{[VIII- 15b]}$$

Or expanding out the cross-products:

$$\begin{aligned}\vec{\mathbf{E}}^{\text{inc}} &= \left\{\mathrm{E}^{\text{inc}}\hat{\mathbf{y}} + \left(\eta_1\,\mathrm{H}^{\text{inc}}_{||}\right)\left[\cos\theta_{\text{inc}}\,\hat{\mathbf{z}} + \sin\theta_{\text{inc}}\,\hat{\mathbf{x}}\right]\right\} \exp\left[-i\,k_1\left(-x\cos\theta_{\text{inc}} + z\sin\theta_{\text{inc}}\right)\right]\\ \vec{\mathbf{H}}^{\text{inc}} &= \left\{\mathrm{H}^{\text{inc}}_{||}\hat{\mathbf{y}} - \left(\eta_1^{-1}\,\mathrm{E}^{\text{inc}}\right)\left[\cos\theta_{\text{inc}}\,\hat{\mathbf{z}} + \sin\theta_{\text{inc}}\,\hat{\mathbf{x}}\right]\right\} \exp\left[-i\,k_1\left(-x\cos\theta_{\text{inc}} + z\sin\theta_{\text{inc}}\right)\right]\end{aligned} \qquad \text{[VIII- 13c]}$$

$$\vec{\mathbf{E}}^{\,\mathrm{ref}} = \left\{ \mathrm{E}^{\mathrm{ref}} \hat{\mathbf{y}} + \left(\eta_1 \mathrm{H}_{||}^{\mathrm{ref}} \right) \left[-\cos\theta_{\mathrm{ref}} \, \hat{\mathbf{z}} + \sin\theta_{\mathrm{ref}} \, \hat{\mathbf{x}} \right] \right\} \exp\left[-i\, k_1 \left(x\cos\theta_{\mathrm{ref}} + z\sin\theta_{\mathrm{ref}} \right) \right]$$
$$\vec{\mathbf{H}}^{\,\mathrm{ref}} = \left\{ \mathrm{H}_{||}^{\mathrm{ref}} \hat{\mathbf{y}} - \left(\eta_1^{-1} \mathrm{E}^{\mathrm{ref}} \right) \left[-\cos\theta_{\mathrm{ref}} \, \hat{\mathbf{z}} + \sin\theta_{\mathrm{ref}} \, \hat{\mathbf{x}} \right] \right\} \exp\left[-i\, k_1 \left(x\cos\theta_{\mathrm{ref}} + z\sin\theta_{\mathrm{ref}} \right) \right] \qquad \text{[VIII- 14c]}$$

$$\vec{\mathbf{E}}^{\,\mathrm{tran}} = \left\{ \mathrm{E}^{\mathrm{tran}} \hat{\mathbf{y}} + \left(\eta_2 \mathrm{H}_{||}^{\mathrm{tran}} \right) \left[\cos\theta_{\mathrm{tran}} \, \hat{\mathbf{z}} + \sin\theta_{\mathrm{tran}} \, \hat{\mathbf{x}} \right] \right\} \exp\left[-i\, k_2 \left(-x\cos\theta_{\mathrm{tran}} + z\sin\theta_{\mathrm{tran}} \right) \right]$$
$$\vec{\mathbf{H}}^{\,\mathrm{tran}} = \left\{ \mathrm{H}_{||}^{\mathrm{tran}} \hat{\mathbf{y}} - \left(\eta_2^{-1} \mathrm{E}^{\mathrm{tran}} \right) \left[\cos\theta_{\mathrm{tran}} \, \hat{\mathbf{z}} + \sin\theta_{\mathrm{tran}} \, \hat{\mathbf{x}} \right] \right\} \exp\left[-i\, k_2 \left(-x\cos\theta_{\mathrm{tran}} + z\sin\theta_{\mathrm{tran}} \right) \right] \qquad \text{[VIII- 15c]}$$

Applying any kind of continuity conditions at the interface requires that

$$\theta_{\mathrm{ref}} = \theta_{\mathrm{inc}} \qquad \textit{Law of Sinus} \qquad \text{[VIII- 16a]}$$

$$k_2 \sin\theta_{\mathrm{tran}} = k_1 \sin\theta_{\mathrm{inc}} \qquad \textit{Law of Snell} \qquad \text{[VIII- 16b]}$$

Applying, in particular, the continuity conditions discussed in the previous section -- *viz.*

$$\left[\vec{\mathbf{E}}^{1} \right]_{\mathrm{tang}} = \left[\vec{\mathbf{E}}^{2} \right]_{\mathrm{tang}} \quad \text{and} \quad \left[\vec{\mathbf{H}}^{1} \right]_{\mathrm{tang}} = \left[\vec{\mathbf{H}}^{2} \right]_{\mathrm{tang}} \qquad \text{[VIII- 17]}$$

at the interface, requires that

$$\mathrm{E}^{\mathrm{inc}} + \mathrm{E}^{\mathrm{ref}} = \mathrm{E}^{\mathrm{tran}}$$
$$\eta_1^{-1} \cos\theta_{\mathrm{inc}} \left[\mathrm{E}^{\mathrm{inc}} - \mathrm{E}^{\mathrm{ref}} \right] = \eta_2^{-1} \cos\theta_{\mathrm{tran}} \left[\mathrm{E}^{\mathrm{tran}} \right] \qquad \text{[VIII- 18]}$$

and that

$$\mathrm{H}_{||}^{\mathrm{inc}} + \mathrm{H}_{||}^{\mathrm{ref}} = \mathrm{H}_{||}^{\mathrm{tran}}$$
$$\eta_1 \cos\theta_{\mathrm{inc}} \left[\mathrm{H}_{||}^{\mathrm{inc}} - \mathrm{H}_{||}^{\mathrm{ref}} \right] = \eta_2 \cos\theta_{\mathrm{tran}} \left[\mathrm{H}_{||}^{\mathrm{tran}} \right] \qquad \text{[VIII- 19]}$$

These two sets of equations yield the **Fresnel Reflection Equations** -- *viz.*

$$\frac{\mathrm{E}^{\mathrm{ref}}}{\mathrm{E}^{\mathrm{inc}}} = \left[\frac{\eta_1^{-1} \cos\theta_{\mathrm{inc}} - \eta_2^{-1} \cos\theta_{\mathrm{tran}}}{\eta_1^{-1} \cos\theta_{\mathrm{inc}} + \eta_2^{-1} \cos\theta_{\mathrm{tran}}} \right] \qquad \text{[VIII- 20a]}$$

and

$$\frac{H_{\|}^{ref}}{H_{\|}^{inc}} = \left[\frac{\eta_1 \cos\theta_{inc} - \eta_2 \cos\theta_{tran}}{\eta_1 \cos\theta_{inc} + \eta_2 \cos\theta_{tran}} \right] \qquad [\text{ VIII- 21a }]$$

Since $\eta_1^{-1} \sin\theta_{inc} = \eta_2^{-1} \sin\theta_{tran}$

$$\frac{E^{ref}}{E^{inc}} = \frac{\cos\theta_{inc} \sin\theta_{tran} - \cos\theta_{tran} \sin\theta_{inc}}{\cos\theta_{inc} \sin\theta_{tran} + \cos\theta_{tran} \sin\theta_{inc}} = \frac{\sin\left(\theta_{tran} - \theta_{inc}\right)}{\sin\left(\theta_{tran} + \theta_{inc}\right)} \qquad [\text{ VIII- 20b }]$$

and

$$\frac{H_{\|}^{ref}}{H_{\|}^{inc}} = \frac{\cos\theta_{inc} \sin\theta_{inc} - \cos\theta_{tran} \sin\theta_{tran}}{\cos\theta_{inc} \sin\theta_{inc} + \cos\theta_{tran} \sin\theta_{tran}} = \frac{\tan\left(\theta_{inc} - \theta_{tran}\right)}{\tan\left(\theta_{inc} + \theta_{tran}\right)} \qquad [\text{ VIII- 21b }]$$

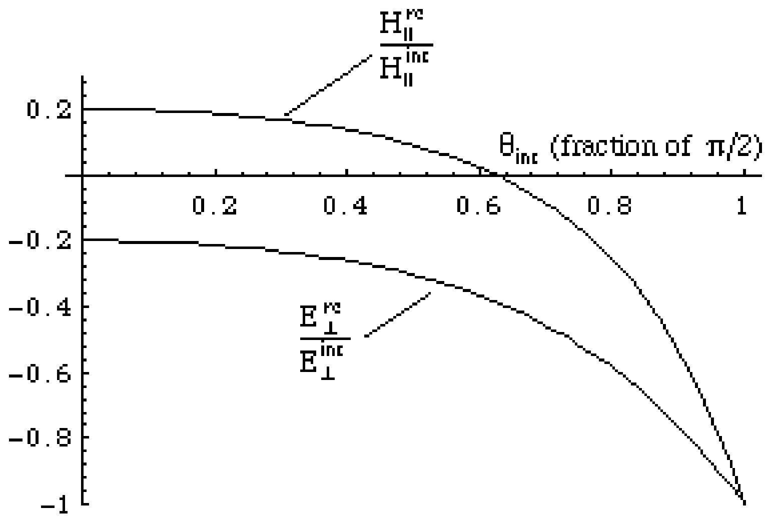
These equations taken together with first equations from Equations [VIII- 18] and [VIII- 19] yield the **Fresnel Transmission Equations** -- *viz*

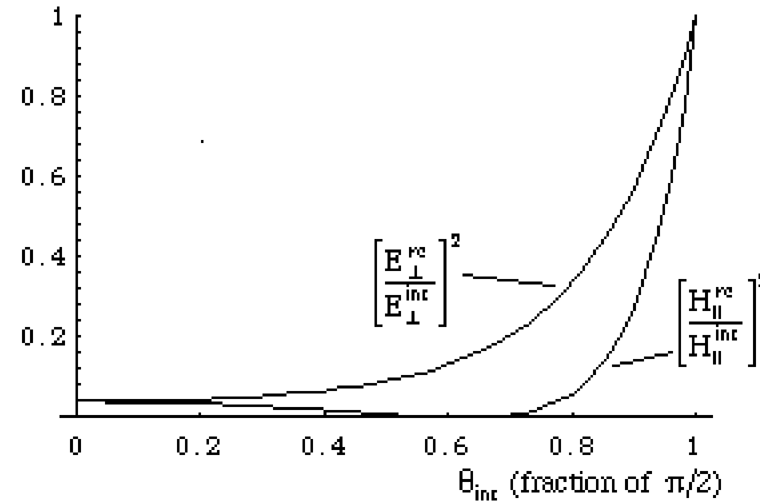
$$\frac{E^{tran}}{E^{inc}} = \frac{2\cos\theta_{inc} \sin\theta_{tran}}{\cos\theta_{inc} \sin\theta_{tran} + \cos\theta_{tran} \sin\theta_{inc}} \qquad [\text{ VIII- 22 }]$$

and

$$\frac{H_{\|}^{tran}}{H_{\|}^{inc}} = \frac{2\cos\theta_{inc} \sin\theta_{inc}}{\cos\theta_{inc} \sin\theta_{inc} + \cos\theta_{tran} \sin\theta_{tran}} \qquad [\text{ VIII- 23 }]$$

FAMOUS FRESNEL REFLECTION CURVES **(** $n_2/n_1 = \eta_1/n_2 = \sqrt{\varepsilon_2/\varepsilon_1} = 1.5$ **)**





The minimum (zero) in $H_{||}^{\mathrm{ref}}/H_{||}^{\mathrm{inc}}$ occurs at the Brewster angle where

$$\tan\left(\theta_{\mathrm{inc}}^{\mathrm{Brewster}} + \theta_{\mathrm{tran}}^{\mathrm{Brewster}}\right) \quad \infty \qquad \text{[VIII- 24a]}$$

or

$$\theta_{\mathrm{tran}}^{\mathrm{Brewster}} = \pi/2 - \theta_{\mathrm{inc}}^{\mathrm{Brewste}} \qquad \text{[VIII- 24b]}$$

or (from Snell's equation)

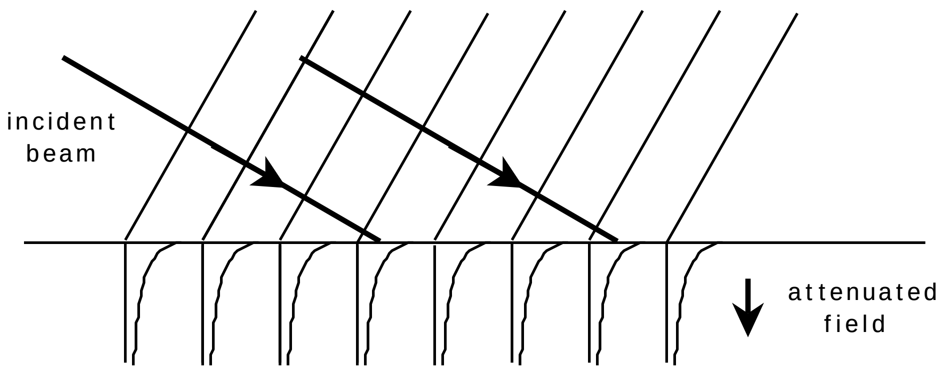
$$\tan\theta_{\mathrm{inc}}^{\mathrm{Brewster}} = \eta_1/\eta_2 = n_2/n_1 = \sqrt{\varepsilon_2/\varepsilon_1} \ . \qquad \text{[VIII- 24c]}$$

Total Internal Reflection

Reconsider Equation [VIII- 15c] and use Snell's law to write the exponential factors in the form

$$\vec{\mathbf{E}}^{\rm tran} = \left\{ \mathrm{E}^{\rm tran}\hat{\mathbf{y}} + \left(\eta_2\, \mathrm{H}_{||}^{\rm tran}\right)\left[\cos\theta_{\rm tran}\,\hat{\mathbf{z}} + \sin\theta_{\rm tran}\,\hat{\mathbf{x}}\right]\right\} \exp\left[i x \sqrt{k_2^2 - k_1^2 \sin^2\theta_{\rm inc}}\right] \exp\left[-i z \; k_1 \sin\theta_{\rm inc}\right] \quad \text{[VIII- 25]}$$

When $\sin\theta_{\rm inc} > k_2/k_1 = n_2/n_1 \equiv \sin\theta_{\rm inc}^{\rm crit}$, $\vec{\mathbf{E}}^{\rm tran}$, the solution in medium 2, is **attenuated**!



Reconsideration of Equation [VIII- 20a] and [VIII- 21a] shows that the magnitude of the reflection coefficients are **one** when $\sin\theta_{\rm inc} > \sin\theta_{\rm inc}^{\rm crit}$ -- *viz*.

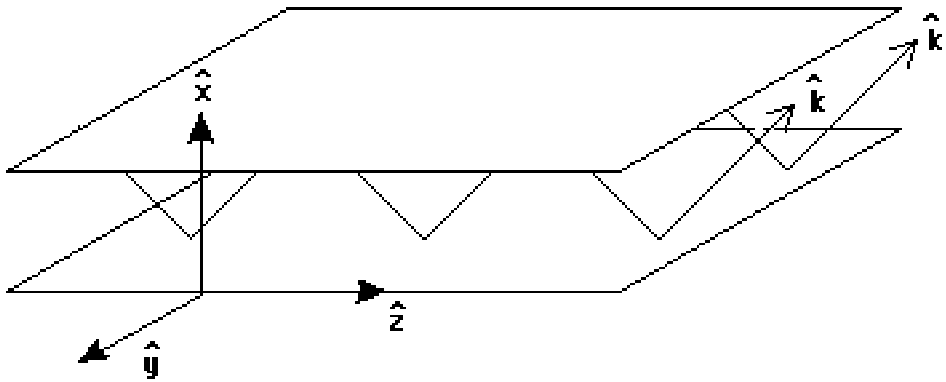
$$\frac{\mathrm{E}^{\rm ref}}{\mathrm{E}^{\rm inc}} = \left[\frac{\cos\theta_{\rm inc} - i\sqrt{\sin^2\theta_{\rm inc} - \left(k_2/k_1\right)^2}}{\cos\theta_{\rm inc} + i\sqrt{\sin^2\theta_{\rm inc} - \left(k_2/k_1\right)^2}}\right] = \exp\left\{-i\, 2\tan^{-1}\left[\frac{\sqrt{\sin^2\theta_{\rm inc} - \left(k_2/k_1\right)^2}}{\cos\theta_{\rm inc}}\right]\right\} \quad \text{[VIII- 26a]}$$

and

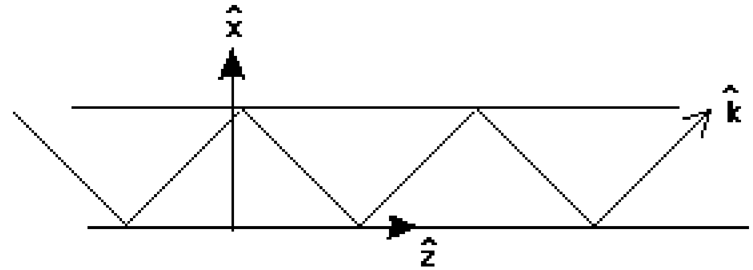
$$\frac{H_{\parallel}^{ref}}{H_{\parallel}^{inc}} = \left[\frac{\cos\theta_{inc} - j\sqrt{\left(k_1/k_2\right)^2 \sin^2\theta_{inc} - 1}}{\cos\theta_{inc} + j\sqrt{\left(k_1/k_2\right)^2 \sin^2\theta_{inc} - 1}}\right] = \exp\left\{-j\, 2\tan^{-1}\left[\frac{\sqrt{\left(k_1/k_2\right)^2 \sin^2\theta_{inc} - 1}}{\cos\theta_{inc}}\right]\right\}. \quad [\text{ VIII- 26b }]$$

PARALLEL PLATE WAVEGUIDE:

Consider the propagation of a plane wave between two parallel perfectly conducting planes.



Perspective view



Side view

Combining Equations [VIII- 13c] and [VIII- 14c], the electric field strength of the TE wave in the region between the plates may be written

$$\vec{\mathbf{E}} = \hat{\mathbf{y}}\left[E^{inc}\exp\left(i x\, k_1 \cos\theta_{inc}\right) + E^{ref}\exp\left(-i x\, k_1 \cos\theta_{inc}\right)\right]\exp\left(-i\, z\, k_1 \sin\theta_{inc}\right) \quad [\text{ VIII-27 }]$$

At $x = 0$ the field parallel to the surface of a perfect conductor must be zero so that $E^{ref} = -E^{inc}$ and, therefore,

$$\begin{aligned}\vec{\mathbf{E}} &= \hat{\mathbf{y}}\, E^{inc}\left[\exp\left(i x\, k_1 \cos\theta_{inc}\right) - \exp\left(-i\, x k_1 \cos\theta_{inc}\right)\right]\exp\left(-i z\, k_1 \sin\theta_{inc}\right) \\ &= \hat{\mathbf{y}}\, 2\, i\, E^{inc}\sin\left(x k_1 \cos\theta_{inc}\right)\exp\left(-i\, z\, \beta\right)\end{aligned} \quad [\text{ VIII-28 }]$$

where $\beta = k_1 \sin\theta_{inc}$. At the upper surface -- *i.e.* $x = d$ -- the field parallel to the surface of a perfect conductor must also be zero so that

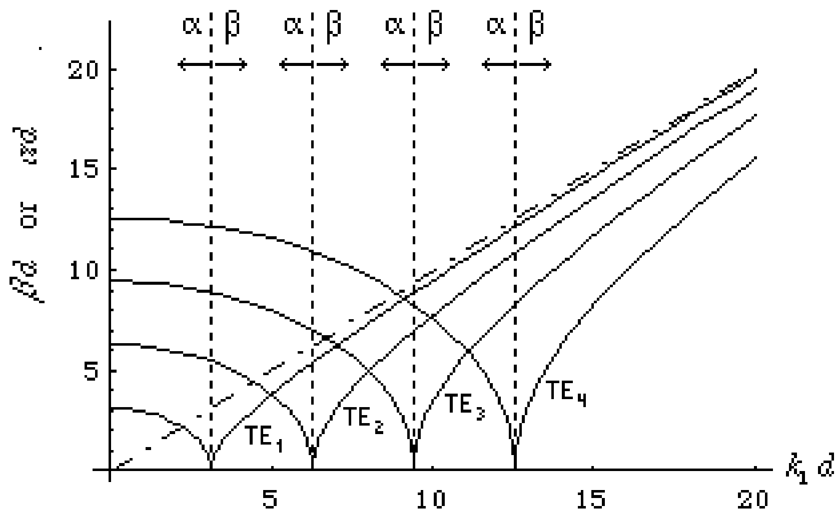
$$d\, k_1 \cos\theta_{inc} = n\,\pi \qquad \text{where } n = 1,2,3,\ldots \qquad [\text{ VIII-29 }]$$

and, therefore,

$$\beta_{TE_n} = k_1 \sin\theta_{inc} = \sqrt{k_1^2 - k_1^2\cos^2\theta_{inc}} = \sqrt{k_1^2 - (n\,\pi/d)^2} \qquad \text{where } n = 1,2,3,\ldots \qquad [\text{ VIII-30 }]$$

which is **the dispersion relationship for TE waves in a parallel plate waveguide** with "cutoff" frequencies at

$$\omega_n^{cutoff} = (n\,\pi/d)\left(\sqrt{\mu_0\,\varepsilon_1}\right)^{-1} \qquad \text{where } n = 1,2,3,\ldots \qquad [\text{ VIII-31 }]$$



Again combining Equations [VIII- 13c] and [VIII- 14c], the electric field strength of the TM wave in the region between the plates may be written

$$\begin{aligned}\vec{\mathbf{E}}_{||} = \hat{\mathbf{z}}\left(\eta_1 \cos\theta_{\text{inc}}\right)\left\{\mathrm{H}_{||}^{\text{inc}} \exp\left(i\,x\,k_1 \cos\theta_{\text{inc}}\right) - \mathrm{H}_{||}^{\text{ref}} \exp\left(-i\,x\,k_1 \cos\theta_{\text{inc}}\right)\right\} \exp\left(-i\,z\,k_1 \sin\theta_{\text{inc}}\right) \\ +\hat{\mathbf{x}}\left(\eta_1 \sin\theta_{\text{inc}}\right)\left\{\mathrm{H}_{||}^{\text{inc}} \exp\left(i\,x\,k_1 \cos\theta_{\text{inc}}\right) + \mathrm{H}_{||}^{\text{ref}} \exp\left(-i\,x\,k_1 \cos\theta_{\text{inc}}\right)\right\} \exp\left(-i\,z\,k_1 \sin\theta_{\text{inc}}\right)\end{aligned} \qquad \text{[VIII-32]}$$

At $x = 0$ the field parallel to the surface of a perfect conductor must be zero so that $\mathrm{H}_{||}^{\text{ref}} = \mathrm{H}_{||}^{\text{inc}}$ and, therefore,

$$\begin{aligned}\vec{\mathbf{E}}_{||} = \hat{\mathbf{z}}\, 2\, i\, \mathrm{H}_{||}^{\text{inc}} \left(\eta_1 \cos\theta_{\text{inc}}\right) \sin\left(x\,k_1 \cos\theta_{\text{inc}}\right) \exp\left(-i z\,\beta\right) \\ +\hat{\mathbf{x}}\, 2\, \mathrm{H}_{||}^{\text{inc}} \left(\eta_1 \sin\theta_{\text{inc}}\right) \cos\left(x\,k_1 \cos\theta_{\text{inc}}\right) \exp\left(-i z\,\beta\right)\end{aligned} \qquad \text{[VIII-33]}$$

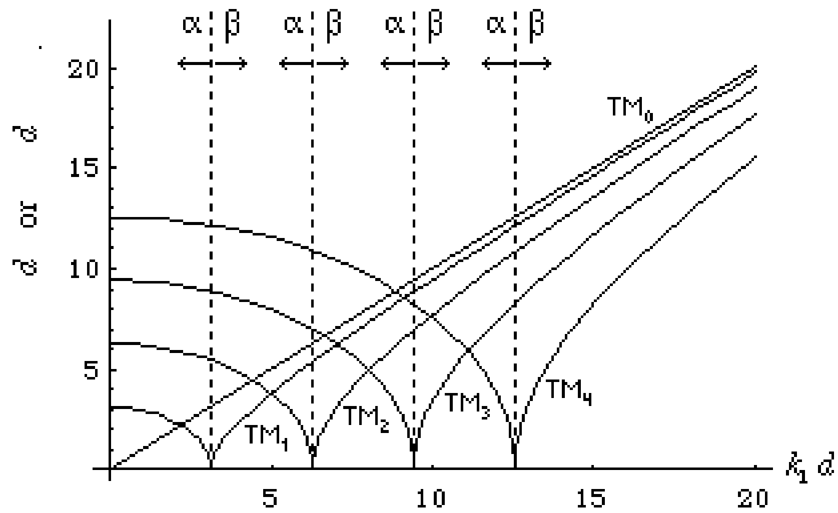
where $\beta = k_1 \sin\theta_{\text{inc}}$. At the upper surface -- *i.e.* $x = d$ -- again the field parallel to the surface of a perfect conductor must also be zero so that

$$d\,k_1 \cos\theta_{\text{inc}} = n\,\pi \qquad \text{where } n = 0,1,2,3,\ldots \qquad \text{[VIII-34]}$$

and, therefore,

$$\beta_{\mathrm{TM}_n} = k_1 \sin\theta_{\text{inc}} = \sqrt{k_1^2 - k_1^2 \cos^2\theta_{\text{inc}}} = \sqrt{k_1^2 - \left(n\,\pi/d\right)^2} \qquad \text{where } n = 0,1,2,3,\ldots \quad \text{[VIII-35]}$$

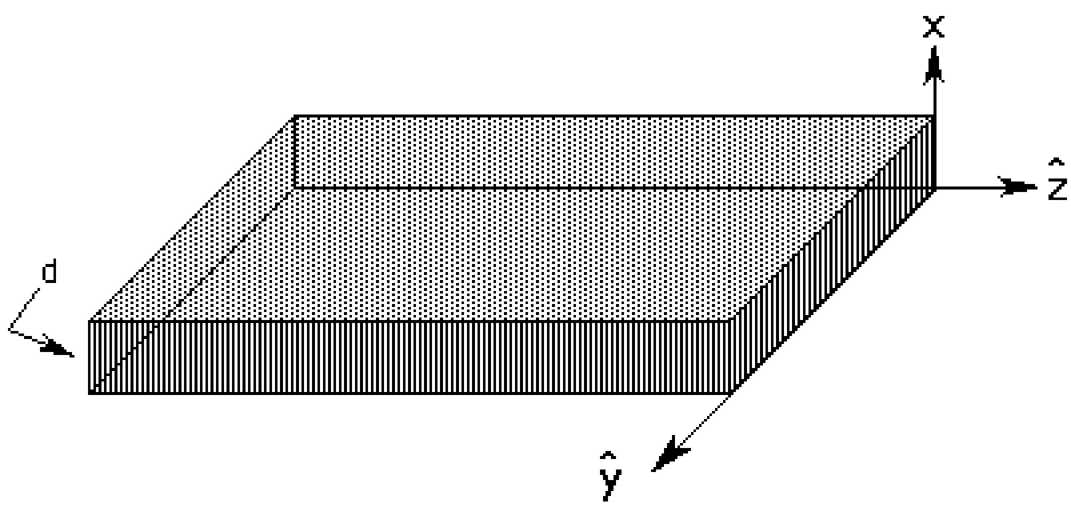
which is **the dispersion relationship for TM waves in a parallel plate waveguide.**



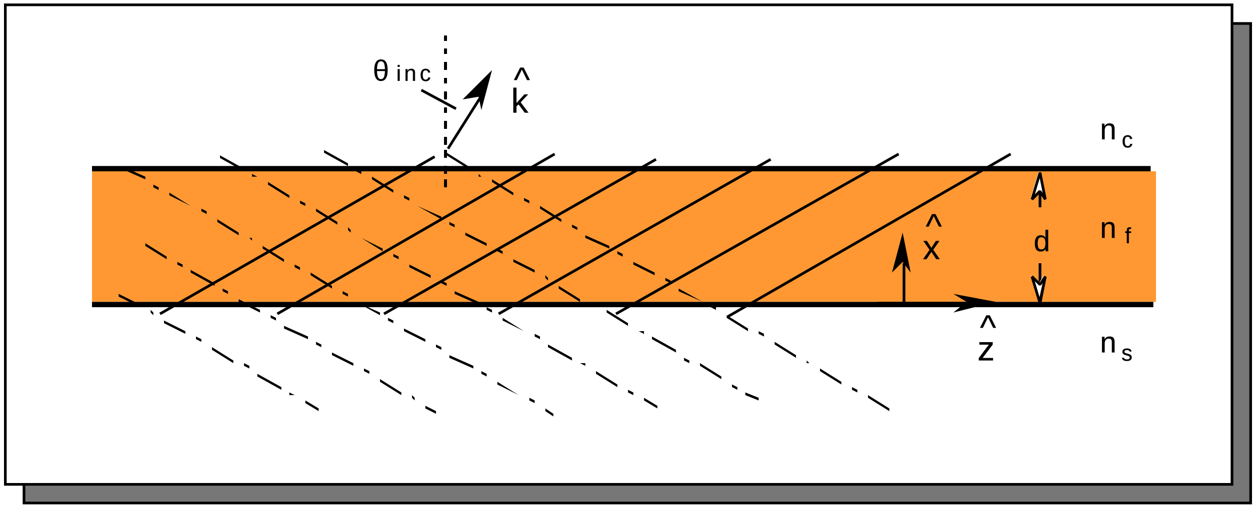
Note that the TM_0 mode is a bona fide mode of propagation which does not have a "cutoff" frequency!

DIELECTRIC SLAB WAVEGUIDE:

Consider the propagation of waves "trap in" or "guided by" a dielectric slab of thickness *d*.



In its full generality this is moderately complicated problem, but a rather simple ray optics model of the propagation is sufficient to yield dispersion relationships for the various possible modes of propagation. To obtain such relationships, consider the total internal reflection of a sequence of plane waves as illustrated below.



In order for the multiplely reflected wave to be **self-consistence** the following, relatively obvious, phase condition must hold:[33]

$$\Delta\varphi_{x=d} + \Delta\varphi_{x=0} + 2\, k_1\, d \cos\theta_{inc} = m\, 2\pi \quad \text{where} \quad m = 0,1,2,3,\ldots \qquad [\text{ VIII- 36 }]$$

where $\Delta\varphi_{x=d}$ and $\Delta\varphi_{x=0}$ are, respectively, the phase shifts associated with the reflections at the upper and lower dielectric boundaries.

For **TE-modes of propagation** Equation [VIII- 26a] gives the phase shift at the boundary (called in the trade *the TE Goos-Hänchen shift*) and Equation [VIII- 36] becomes

33 This equation is a direct generalization of Equations [V-29] and [V-34] which figured in our analysis of parallel plane waveguides.

$$k_f\, d\cos\theta_{inc} = m\,\pi + \tan^{-1}\left[\frac{\sqrt{\sin^2\theta_{inc} - \left(k_c/k_f\right)^2}}{\cos\theta_{inc}}\right] + \tan^{-1}\left[\frac{\sqrt{\sin^2\theta_{inc} - \left(k_s/k_f\right)^2}}{\cos\theta_{inc}}\right] \qquad \text{[VIII- 37a]}$$

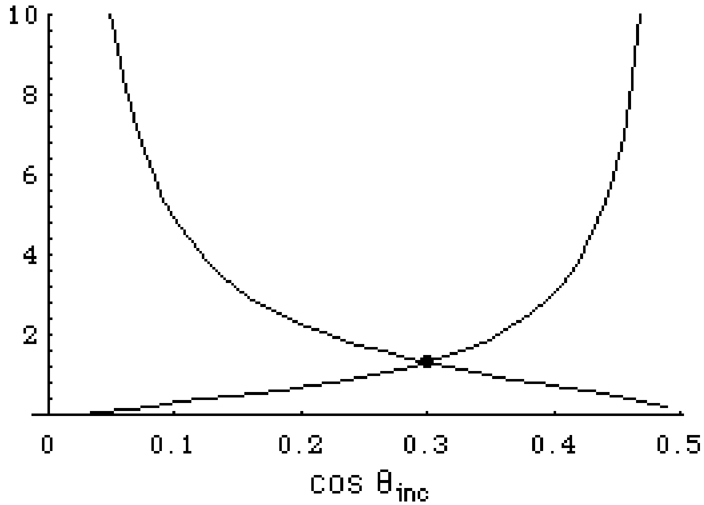
or

$$\left(\frac{2\pi d}{\lambda_0}\right)\sqrt{n_f^2 - n^2} = m\,\pi + \tan^{-1}\left[\frac{\sqrt{n^2 - n_c^2}}{\sqrt{n_f^2 - n^2}}\right] + \tan^{-1}\left[\frac{\sqrt{n^2 - n_s^2}}{\sqrt{n_f^2 - n^2}}\right] \qquad \text{[VIII- 37b]}$$

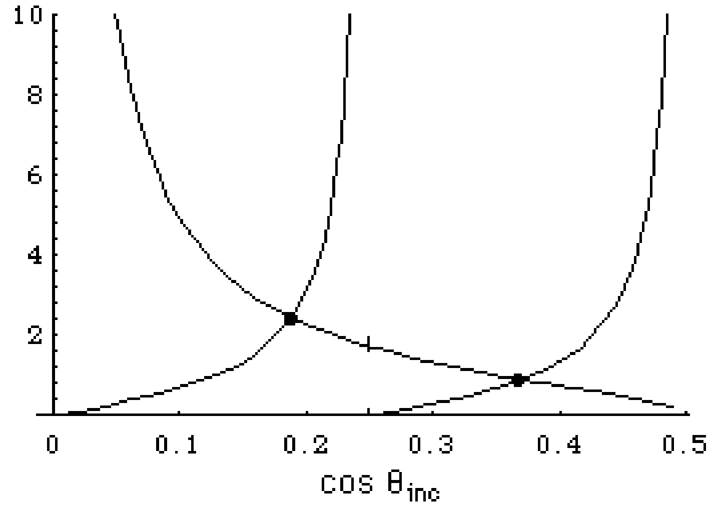
where $\sin\theta_{inc} \equiv \beta/k_f = n/n_f$ (***n* is the effective index of the propagation mode**). For the symmetric case (i.e., $n_c \equiv n_s$), the **self-consistence relationship for the TE modes** is given by

$$\frac{\sqrt{\sin^2\theta_{int} - \left(n_s/n_f\right)^2}}{\cos\theta_{int}} = \tan\left[\frac{n_f\, k_0\, d\, \cos\theta_{int}}{2} - m\,\frac{\pi}{2}\right] \qquad \text{[VIII- 38]}$$

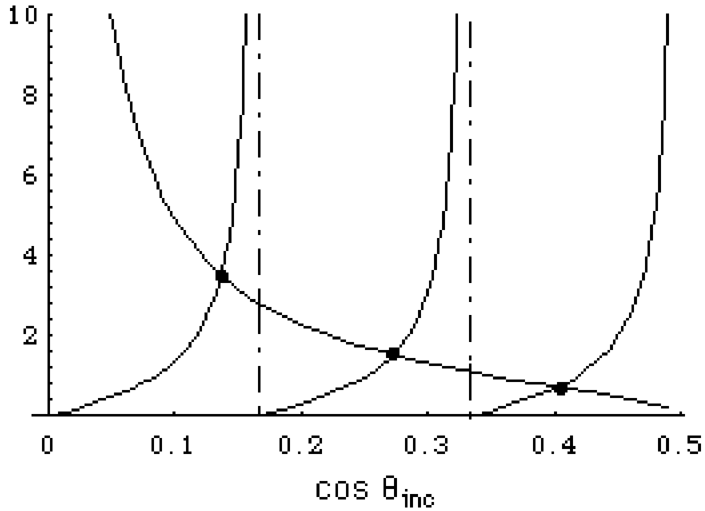
where $k_0 = \omega/c = 2\pi/\lambda_0$. This is a transcendental equation in the single variable $\cos\theta_{inc}$. Its solutions yield the allowed bounce angles, $\left(\theta_{inc}\right)_m$, of possible modes and, hence, the allowed propagation constants since $\beta = k_f\, \sin\theta_{inc}$ The left and right sides of this equation may be plot as a function of $\cos\theta_{inc}$ with $n_f\, k_0\, d = n_f\, 2\pi\left(d/\lambda_0\right)$ and $\sin\theta_{inc}^{crit} = n_s/n_f$ as a parameters. The intersections of such curves yield the allowed bounce angles as illustrated below



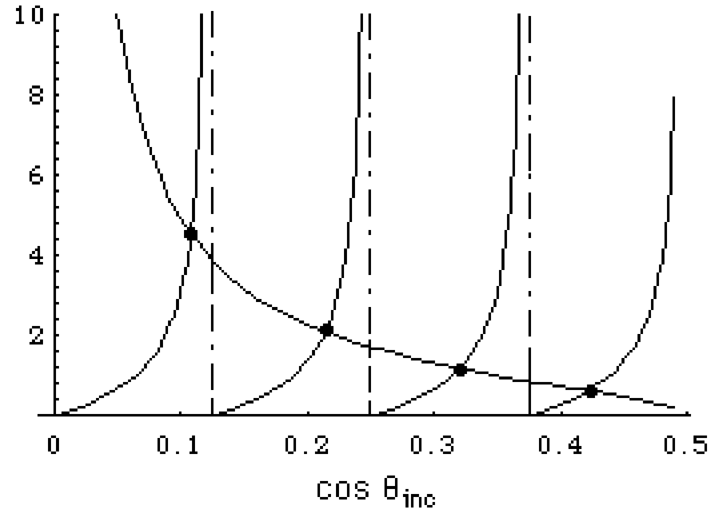
LHS and RHS of Equation [VIII- 38]
for $n_1\left(d/\lambda_0\right) = 0.5$ and $\cos\theta_{\rm inc}^{\rm crit} = 0.5$



LHS and RHS of Equation [VIII- 38]
for $n_1\left(d/\lambda_0\right) = 1.0$ and $\cos\theta_{\rm inc}^{\rm crit} = 0.5$



LHS and RHS of Equation [VIII- 38]
for $n_1\left(d/\lambda_0\right) = 1.5$ and $\cos\theta_{\rm inc}^{\rm crit} = 0.5$



LHS and RHS of Equation [VIII- 38]
for $n_1\left(d/\lambda_0\right) = 2.0$ and $\cos\theta_{\rm inc}^{\rm crit} = 0.5$

1974, Kogelnik and Ramaswamy [34] developed a convenient formalism for treating slab-waveguide problems. First they introduced three new waveguide parameter -- *viz*.

The normalized frequency/slab thickness parameter $V \equiv k_0\, d\, \sqrt{n_{\mathrm{f}}^2 - n_{\mathrm{s}}^2}$ [VIII-39a]

The normalized waveguide index parameter $b \equiv \left(n^2 - n_{\mathrm{f}}^2\right) / \left(n_{\mathrm{f}}^2 - n_{\mathrm{s}}^2\right)$ [VIII-39b]

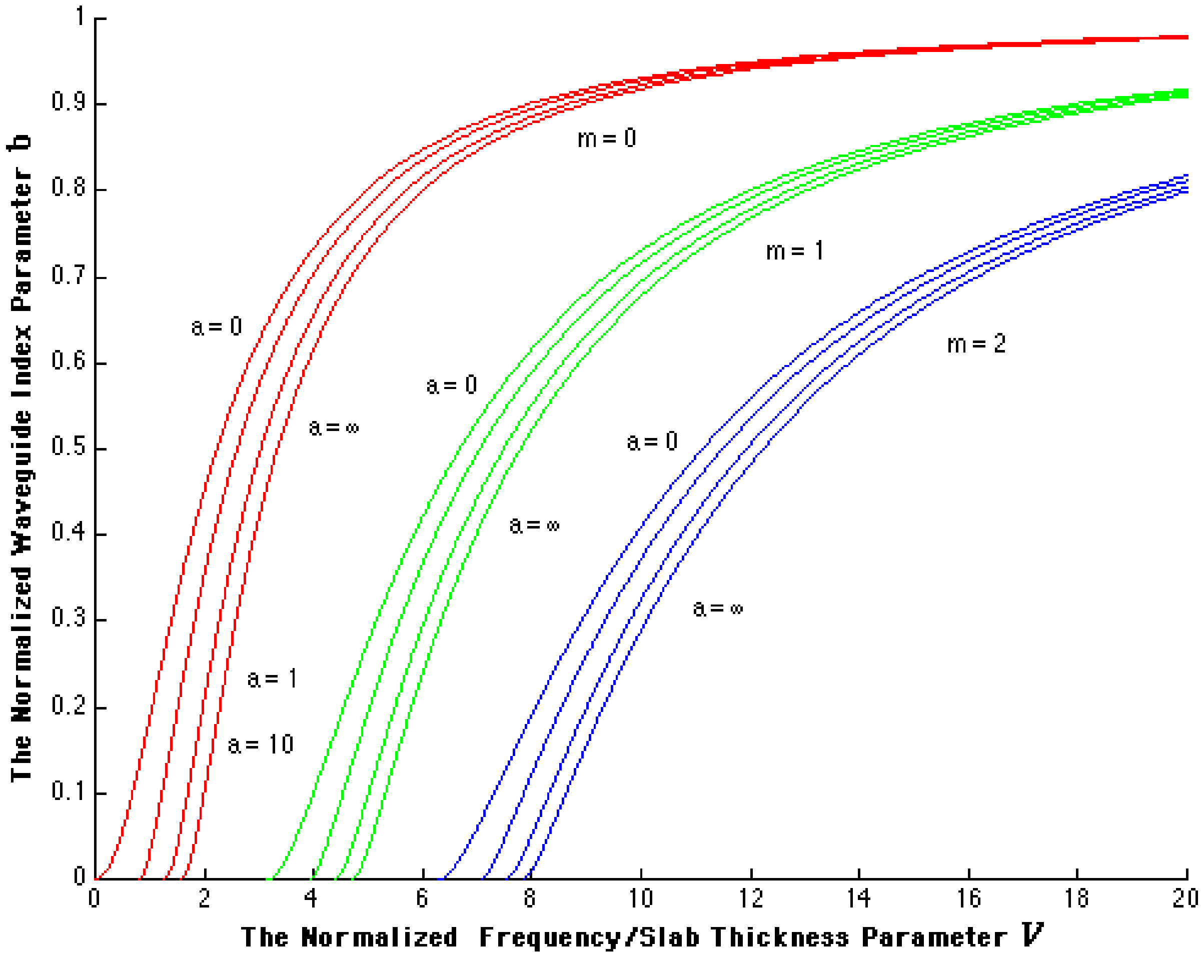
The normalized waveguide asymmetry parameter $a \equiv \left(n_{\mathrm{s}}^2 - n_{\mathrm{c}}^2\right) / \left(n_{\mathrm{f}}^2 - n_{\mathrm{s}}^2\right)$ [VIII-39c]

They then showed that Equation [VIII-37b] could be written

$$V\sqrt{1-b} = m\,\pi + \tan^{-1}\sqrt{b/(1-b)} + \tan^{-1}\sqrt{(a+b)/(1-b)} \qquad \text{[VIII-39c]}$$

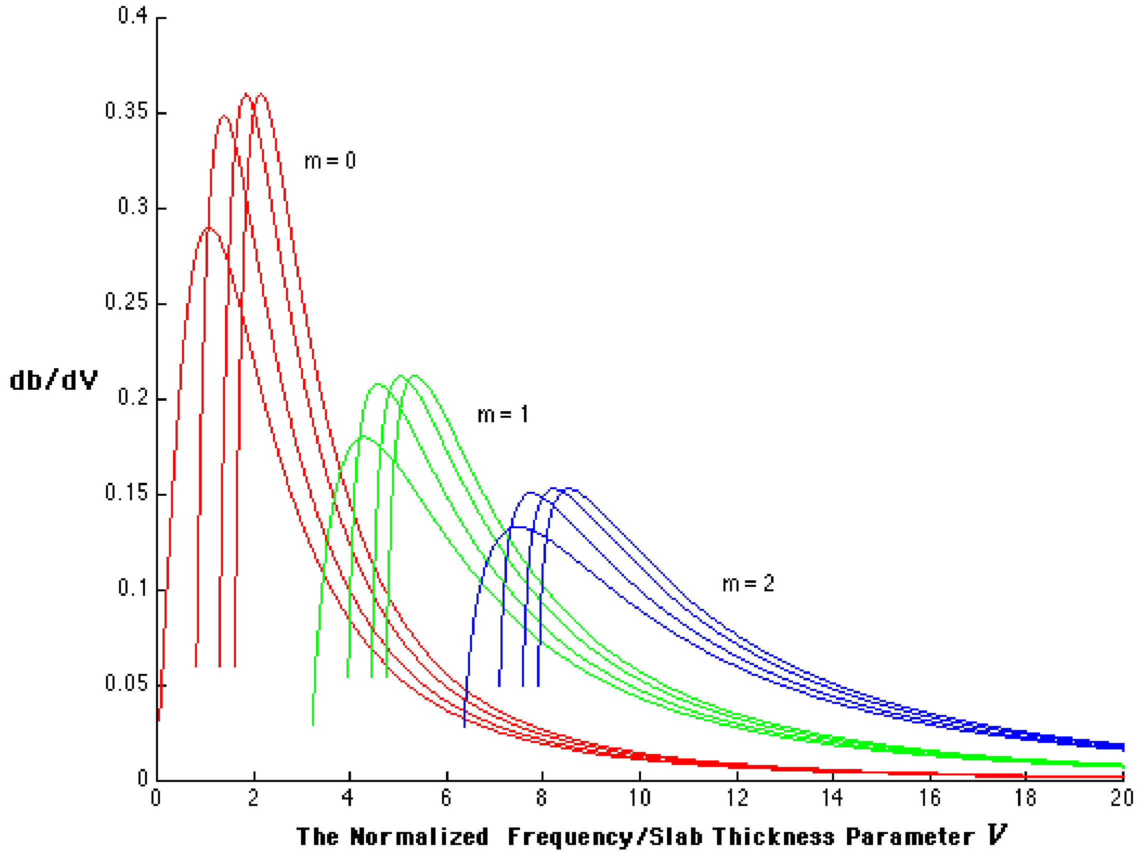
which can be used to generate the following family of curves:

[34] H. Kogelnik and V. Ramaswamy, Appl. Opt. **13**, 1857 (1974).



It is also useful to differentiate Equation [VIII-39c] to obtain

$$\frac{db}{dV} = 2\left(1-b\right)\left[\sqrt{1/b}+\sqrt{1/\left(a+b\right)}\right]^{-1} \qquad \text{[VIII-40]}$$


db/dV
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
m = 0
m = 1
m = 2
0
2
4
6
8
10
12
14
16
18
20
The Normalized Frequency/Slab Thickness Parameter V

IX. OPTICAL PULSE PROPAGATION

THE ELECTROMAGNETIC NONLINEAR SCHRÖDINGER EQUATION:

We begin our discussion of optical pulse propagation[35] with a derivation of the nonlinear Schrödinger (NLS) equation. To that end, we recall Equations [VII-23] and [VII-23] from the early lecture set entitled *Nonlinear Optics I -- i.e.*

$$\nabla^2 \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right) + \left(\omega^2/c^2\varepsilon_0\right)\bar{\bar{\varepsilon}}(\omega)\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = -\mu_0\,\omega^2\,\vec{\mathbf{P}}^{(\mathrm{NL})}(\vec{\mathbf{r}},\omega) \qquad \text{[IX-1]}$$

$$\vec{\nabla}\cdot\left[\bar{\bar{\varepsilon}}(\omega)\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right] = -\vec{\nabla}\cdot\vec{\mathbf{P}}^{(\mathrm{NL})}(\vec{\mathbf{r}},\omega) \qquad \text{[IX-2]}$$

In this treatment we will confine our attention to wave propagation in **uniform, isotropic optical materials** -- *viz.*, glass fibers. For such materials, we can write

$$\vec{\mathbf{P}}^{(\mathrm{NL})}(\vec{\mathbf{r}},\omega) = \varepsilon_{\mathrm{NL}}(\vec{\mathbf{r}},\omega)\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[IX-3]}$$

where $\varepsilon_{\mathrm{NL}}(\vec{\mathbf{r}},\omega) = \frac{3}{4}\varepsilon_0\,\chi^{(3)}_{xxxx}\left|\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right|^2$ and, thus, Equation [IX-1] simplifies to

$$\nabla^2 \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) + \left(k_0^2/\varepsilon_0\right)\left[\varepsilon(\omega) + \varepsilon_{\mathrm{NL}}(\vec{\mathbf{r}},\omega)\right]\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[IX-4]}$$

where $k_0 = \omega/c$.[36]

To proceed, postulate that this nonlinear Helmholtz equation can be treated by ***separation of variables*** methods. In particular, we are looking for a time-localized solution (a

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[35] An excellent reference on this subject is Govind P. Agrawal's *Nonlinear Fiber Optics*, Academic Press (1989) ISBN 0-12-045140-9.

[36] In this simplification, we have taken $\vec{\nabla}\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = -\left[\varepsilon(\omega) - \varepsilon_{\mathrm{NL}}(\vec{\mathbf{r}},\omega)\right]^{-1}\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\cdot\vec{\nabla}\varepsilon_{\mathrm{NL}}(\vec{\mathbf{r}},\omega) \approx 0$.

pulse) with a relatively narrow frequency spectrum (or "group" of frequencies centered on a frequency ω_{ctr}. Thus, we assume a seoaration of variables solution

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \mathrm{F}(x,y)\ \vec{\mathbf{G}}(z,\omega-\omega_{ctr})\ \exp\left(-\,i\,\beta_{ctr}\,z\right) \qquad [\text{ IX-5 }]$$

where β_{ctr} is a wave or propagation number to be associated with ω_{ctr} and, thus, Equation [IX-4] becomes

$$\left\{\frac{\partial^2 \mathrm{F}}{\partial x^2}+\frac{\partial^2 \mathrm{F}}{\partial y^2}+\left[k^2-\overline{\beta}^2\right]\ \mathrm{F}\right\}\vec{\mathbf{G}}+\left\{\frac{\partial^2\vec{\mathbf{G}}}{\partial z^2}-i\ 2\beta_{ctr}\frac{\partial\vec{\mathbf{G}}}{\partial z}+\left[\overline{\beta}^2-\beta_{ctr}^2\right]\ \vec{\mathbf{G}}\right\}\mathrm{F}=0 \qquad [\text{ IX-6 }]$$

where $k = k_0\sqrt{\left[\varepsilon(\omega)+\varepsilon_{NL}(\vec{\mathbf{r}},\omega)\right]/\varepsilon_0}$. In the linear problem $\overline{\beta}^2$ would be the "separation constant," but in this case we will need a bit more elaboration. Nevertheless, we shall assume that we can find a set of functions $\mathrm{F}(x,y)$ and values $\overline{\beta}^2$ that satisfy the equation

$$\frac{\partial^2 \mathrm{F}}{\partial x^2}+\frac{\partial^2 \mathrm{F}}{\partial y^2}+\left[k^2-\overline{\beta}^2\right]\ \mathrm{F}=0 \qquad [\text{ IX-7a }]$$

so that

$$\frac{\partial^2\vec{\mathbf{G}}}{\partial z^2}-i\ 2\beta_{ctr}\frac{\partial\vec{\mathbf{G}}}{\partial z}+\left[\overline{\beta}^2-\beta_{ctr}^2\right]\ \vec{\mathbf{G}}=0 \qquad [\text{ IX-7b }]$$

To use perturbation theory, we first reduce Equation [IX-7a] to a solvable linear problem by writing

$$\begin{aligned} k^2 &= k_0^2\left[\varepsilon(\omega)+\varepsilon_{NL}(\vec{\mathbf{r}},\omega)\right]/\varepsilon_0=\left[n(\omega)+\Delta\,n(\omega)\right]^2\ k_0^2 \\ &\approx\left[n^2(\omega)+2\,n(\omega)\ \Delta\,n(\omega)\right]\ k_0^2 \end{aligned} \qquad [\text{ IX-8a }]$$

$$\overline{\beta}^2=\left[\beta+\Delta\beta\right]^2\approx\beta^2+2\beta\ \Delta\beta \qquad [\text{ IX-8b }]$$

where $\Delta n(\omega) = \dfrac{\varepsilon_{\mathrm{NL}}(\vec{\mathbf{r}},\omega)}{2\,n(\omega)} = \dfrac{3}{8}\,\dfrac{\chi^{(3)}_{xxxx}(\omega)}{n(\omega)}\left|\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right|^2$. Thus, to first order we need to solve the linear equation

$$\frac{\partial^2 \mathrm{F}}{\partial x^2} + \frac{\partial^2 \mathrm{F}}{\partial y^2} + \left[n^2\; k_0^2 - \beta^2\right] \mathrm{F} = 0 \; . \qquad [\text{ IX-9 }]$$

which taken together with appropriate boundary conditions defines the linear eigenvalue problem for propagation in the medium where the functions F are the eigenfunctions and the values β are the eigenfunctions. In the previous lecture set -- *i.e.,* VIII. Guided Waves in Planar Structures -- we found a set of eigenfunctions and eigenvalues appropriate to the dielectric-slab guidewave propagation

Following an earlier discussion, we now presume that $\left|\dfrac{\partial^2\vec{\mathbf{G}}}{\partial z^2}\right| \ll \left|2\beta_{\mathrm{ctr}}\dfrac{\partial\vec{\mathbf{G}}}{\partial z}\right|$ -- *i.e.*, we take the ***slowly vary amplitude*** or ***paraxial*** approximation -- so that Equation [IX-7b] reduces to

$$-i\; 2\beta_{\mathrm{ctr}}\frac{\partial\vec{\mathbf{G}}}{\partial z} + \left[\overline{\beta}^{\,2} - \beta_{\mathrm{ctr}}^2\right]\vec{\mathbf{G}} = 0 \qquad [\text{ IX-10a }]$$

where $\overline{\beta}^{\,2} - \beta_{\mathrm{ctr}}^2 \approx \beta^2 - \beta_{\mathrm{ctr}}^2 + 2\,\beta\,\Delta\beta$ and, thus,

$$-i\; 2\beta_{\mathrm{ctr}}\frac{\partial\vec{\mathbf{G}}}{\partial z} + \left[\beta^2 - \beta_{\mathrm{ctr}}^2\right]\vec{\mathbf{G}} + 2\,\beta\,\Delta\beta\,\vec{\mathbf{G}} = 0 \qquad [\text{ IX-10b }]$$

For equation [IX-7a] to be completely satisfied in first order, we must have

$$\Delta\beta = \frac{n\,k_0^2}{\beta}\,\frac{\iint \Delta n\,\left|\mathrm{F}\left(x,y\right)\right|^2\,dx\,dy}{\iint \left|\mathrm{F}\left(x,y\right)\right|^2\,dx\,dy}$$

$$= \frac{3\,k_0^2}{8\,\beta}\,\chi^{(3)}_{xxxx}\left|\vec{\mathbf{G}}\left(z,\omega-\omega_{\mathrm{ctr}}\right)\right|^2\,\frac{\iint \left|\mathrm{F}\left(x,y\right)\right|^4\,dxdy}{\iint \left|\mathrm{F}\left(x,y\right)\right|^2\,dxdy} \qquad \text{[IX-11]}$$

Since we are assuming that propagating pulse has a relatively narrow frequency spectrum, it is reasonable to use a Taylor expansion around ω_{ctr} for $\beta\left(\omega\right)$ -- *viz.*

$$\beta\left(\omega\right) \approx \beta_{\mathrm{ctr}} + \left(\omega-\omega_{\mathrm{ctr}}\right)\beta_1 + \frac{1}{2!}\left(\omega-\omega_{\mathrm{ctr}}\right)^2\beta_2 + \frac{1}{3!}\left(\omega-\omega_{\mathrm{ctr}}\right)^3\beta_3 + \ldots \qquad \text{[IX-12]}$$

where $\beta_\ell = \left.\dfrac{d^\ell\beta\left(\omega\right)}{d\omega^\ell}\right|_{\omega=\omega_c}$.

Near the **dispersion minimum** in glass fibers (*i.e.* $\lambda \approx 1.3-1.6\,\mu m$) we may, to very good approximation, stop with the quadratic term and write

$$\beta^2 - \beta_{\mathrm{ctr}}^2 \approx 2\beta_{\mathrm{ctr}}\left[\beta-\beta_{\mathrm{ctr}}\right] = 2\,\beta_{\mathrm{ctr}}\left[\left(\omega-\omega_{\mathrm{ctr}}\right)\beta_1 + \frac{1}{2!}\left(\omega-\omega_{\mathrm{ctr}}\right)^2\beta_2\right]. \qquad \text{[IX-13]}$$

In this approximation, Equation [IX-10] becomes

$$\frac{\partial\vec{\mathbf{G}}}{\partial z} + i\left[\left(\omega-\omega_{\mathrm{ctr}}\right)\beta_1 + \frac{1}{2!}\left(\omega-\omega_{\mathrm{ctr}}\right)^2\beta_2\right]\vec{\mathbf{G}} + \frac{i\,\beta}{\beta_{\mathrm{ctr}}}\,\Delta\beta\,\vec{\mathbf{G}} = 0\ . \qquad \text{[IX-14a]}$$

If we take

$$\gamma = \frac{\beta}{\beta_{\mathrm{ctr}}}\,\Delta\beta = \frac{3\,k_0^2}{8\beta_{\mathrm{ctr}}}\,\chi^{(3)}_{xxxx}\,\frac{\iint \left|\mathrm{F}\left(x,y\right)\right|^4\,dxdy}{\iint \left|\mathrm{F}\left(x,y\right)\right|^2\,dxdy} \qquad \text{[IX-15]}$$

Equation [IX-10] in the **frequency domain** reduces to

$$\frac{\partial \vec{\mathbf{G}}}{\partial z} + i\left[(\omega-\omega_c)\,\beta_1 + \frac{1}{2!}(\omega-\omega_c)^2\,\beta_2\right]\vec{\mathbf{G}} = -i\,\gamma\left|\vec{\mathbf{G}}\right|^2\vec{\mathbf{G}}\ . \qquad \text{[IX-14b]}$$

which implies the following **time domain equation** for the pulse envelope:

$$\frac{\partial \vec{\mathbf{G}}(z,t)}{\partial z} + \beta_1 \frac{\partial \vec{\mathbf{G}}(z,t)}{\partial t} - \frac{i}{2}\beta_2 \frac{\partial^2 \vec{\mathbf{G}}(z,t)}{\partial t^2} + \frac{\alpha}{2}\vec{\mathbf{G}}(z,t) = -i\,\gamma\left|\vec{\mathbf{G}}(z,t)\right|^2 \vec{\mathbf{G}}(z,t) \qquad \text{[IX-16]}$$

The last term on the left hand side has been added to incorporate the effects of various possible loss mechanisms.

Next we transform into a coordinate system which moves with the "group" -- what might be called the ***surfer's coordinates*** of the pulse -- *i.e.*, $\{t,z\} \quad \{\tau,\varsigma\}$ where $\tau = t - t_0 - \beta_1 (z - z_0)$ and $\varsigma = z - z_0 = z - \beta_1^{-1} t_0$. In term of these surfer's coordinates, Equation [IX-16] becomes[37]

$$\frac{\partial \vec{\mathbf{G}}(\varsigma,\tau)}{\partial \varsigma} = -\frac{\alpha}{2}\vec{\mathbf{G}}(\varsigma,\tau) + \frac{i}{2}\beta_2 \frac{\partial^2 \vec{\mathbf{G}}(\varsigma,\tau)}{\partial \tau^2} - i\,\gamma\left|\vec{\mathbf{G}}(\varsigma,\tau)\right|^2 \vec{\mathbf{G}}(\varsigma,\tau) \qquad \text{[IX-17]}$$

Obviously, if we omit all of the terms on the right hand side of this equation so that $\frac{\partial \vec{\mathbf{G}}(\varsigma,\tau)}{\partial \varsigma} = 0$, we would have the ideal situation wherein a **pulse of any shape propagates forever without changing shape at a velocity** $v_g = \beta_1^{-1}$ --the group

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37 Since

$$\frac{\partial}{\partial z}\vec{\mathbf{G}}(z,t) = \frac{\partial}{\partial \varsigma}\vec{\mathbf{G}}(\varsigma,\tau)\frac{\partial \varsigma}{\partial z} + \frac{\partial}{\partial \tau}\vec{\mathbf{G}}(\varsigma,\tau)\frac{\partial \tau}{\partial z} = \frac{\partial}{\partial \varsigma}\vec{\mathbf{G}}(\varsigma,\tau)\,(1) + \frac{\partial}{\partial \tau}\vec{\mathbf{G}}(\varsigma,\tau)\left(-v_g^{-1}\right)$$

$$\frac{\partial}{\partial t}\vec{\mathbf{G}}(z,t) = \frac{\partial}{\partial \varsigma}\vec{\mathbf{G}}(\varsigma,\tau)\frac{\partial \varsigma}{\partial t} + \frac{\partial}{\partial \tau}\vec{\mathbf{G}}(\varsigma,\tau)\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial \varsigma}\vec{\mathbf{G}}(\varsigma,\tau)\,(0) + \frac{\partial}{\partial \tau}\vec{\mathbf{G}}(\varsigma,\tau)\,(1)\ .$$

velocity. In treating the less-than-ideal situation, we will initially neglect the effects of the loss (first) term and confine our attention to the competing effects of the **dispersion** (second) and **nonlinear** (third) term . We cast the lossless version of Equation [IX-16] into a normalized, standard form by introducing

$$u(Z,T) = \frac{\vec{\mathbf{G}}(\varsigma,\tau)}{\sqrt{P_0}}, \qquad Z = \frac{\varsigma}{L_{\mathrm{D}}} = \varsigma\frac{|\,\beta_2\,|}{\Delta\tau^2}, \qquad T = \sqrt{2}\,\frac{\tau}{\Delta\,\tau}$$

where $\Delta\tau$ is the width of the pulse and P_0 is its peak power.

Thus, we, at last, obtain the standard form of the **nonlinear Schrödinger (NLS) equation**

$$i\frac{\partial u(Z,T)}{\partial Z} - \mathrm{sgn}(\beta_2)\,\frac{\partial^2 u(Z,T)}{\partial T^2} + N^2\left|u(Z,T)\right|^2 u(Z,T) = 0 \qquad \text{[IX-18]}$$

where $N^2 = L_{\mathrm{D}}/L_{\mathrm{NL}} = \gamma\,P_0\,\Delta\tau^2/|\,\beta_2\,|$.

<u>PULSE SOLUTIONS OF LINEAR SCHRÖDINGER EQUATION</u>

If in Equation [IX-10] we set $\vec{\mathbf{G}}(z,t) \quad U(z,t)$ and neglect the loss and nonlinear terms, we see that $\mathrm{U}(z,t)$ satisfies the following differential equation:

$$\frac{\partial}{\partial z}\mathrm{U}(z,t) + v_g^{-1}\frac{\partial}{\partial t}\mathrm{U}(z,t) = i\,b\,\frac{\partial^2}{\partial t^2}\mathrm{U}(z,t) \qquad \text{[IX-19]}$$

For **minimal dispersion** -- *viz.* if $b = \frac{1}{2!}\frac{d^2\beta(\omega)}{d\omega^2}\bigg|_{\omega=\omega_0} \approx 0$ -- Equation [IX-19] becomes

$$\frac{\partial}{\partial z}\mathrm{U}(z,t) + v_g^{-1}\frac{\partial}{\partial t}\mathrm{U}(z,t) = 0 \qquad \text{[IX-20]}$$

which is the basic wave equation for **minimally dispersive media** with the general solution

$$U(z,t) = U(z - v_g\, t) \qquad [\text{ IX-21 }]$$

where $v_g = \left[\left. \frac{d\,\beta(\omega)}{d\omega} \right|_{\omega=\omega_0} \right]^{-1}$ is the **group velocity** of the pulse.

When limitation to **first order dispersion** is an adequate approximation, Equation [IX-19] expressed in surfer's coordinates becomes

$$b\,\frac{\partial^2}{\partial\tau^2} U(\varsigma,\tau) + i\,\frac{\partial}{\partial\varsigma} U(\varsigma,\tau) = 0 \qquad [\text{ IX-22 }]$$

where $b = \beta_2/2$.

Amazingly, this pulse dispersion equation is the, so called, parabolic equation that we saw earlier in connection with beam propagation -- *viz.* the paraxial wave propagation equation.

<u>Solution of Pulse Dispersion Equation</u>

For convenience, we restate here Equation [IX-22 } the first-order pulse dispersion equation -- *viz.*

$$b\,\frac{\partial^2}{\partial\tau^2} U(\varsigma,\tau) + i\,\frac{\partial}{\partial\varsigma} U(\varsigma,\tau) = 0\ .$$

Let us write a Fourier transform for this modulation in terms of "surfer time" -- *i.e.*

$$\mathrm{U}(\varsigma,\tau)=\int_{-\infty}^{+\infty}\mathrm{U}(\varsigma,\omega)\ \exp(i\,\omega\,\tau)\ d\omega \qquad \text{[IX-23a]}$$

where

$$\mathrm{U}(\varsigma,\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{U}(\varsigma,\tau)\ \exp(-i\,\omega\,\tau)\ d\tau\ . \qquad \text{[IX-23b]}$$

Thus, the pulse dispersion equation reduces to an ordinary differential equation -- *viz.*

$$i\,\frac{d}{d\varsigma}\,\mathrm{U}(\varsigma,\omega)=b\ \omega^2\ \mathrm{U}(\varsigma,\omega) \qquad \text{[IX-24]}$$

for the Fourier transform and thus we have the simple solution

$$\mathrm{U}(\varsigma,\omega)=\mathrm{U}(0,\omega)\ \exp\left(-i\ b\,\omega^2\ \varsigma\right)\ . \qquad \text{[IX-25]}$$

Thus, we see that the dispersion changes the phase of **each spectral component** of the pulse by an amount that depends on the frequency and the propagated distance. The general solution may be written as

$$\mathrm{U}(\varsigma,\tau)=\int_{-\infty}^{+\infty}\mathrm{U}(0,\omega)\ \exp\left[i\left(\omega\,\tau-\ b\ \omega^2\ \varsigma\right)\right]\ d\omega \qquad \text{[IX-26a]}$$

where

$$\mathrm{U}(0,\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{U}(0,t-t_0)\ \exp\left[-i\,\omega\left(t-t_0\right)\right]\ d\left(t-t_0\right)\ . \qquad \text{[IX-26b]}$$

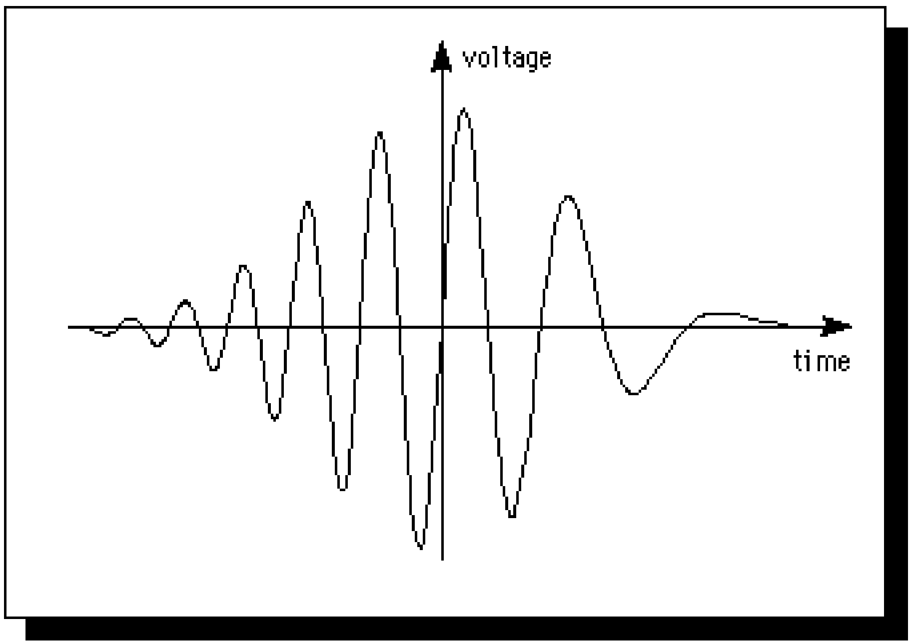
Let us suppose that we have a "chirped" Gaussian at $\varsigma=0$ -- *i.e.*

$$\mathrm{U}(0, t - t_0) = \exp\left[-\left(\frac{1 - iC}{2} \right) \left(\frac{t - t_0}{\tau_0} \right)^2 \right] \qquad [\text{ IX-27a }]$$

so that

$$\mathrm{U}(0, \omega) = \sqrt{\frac{\tau_0^2}{2\pi(1 - iC)}} \exp\left[-\frac{\tau_0^2 \, \omega^2}{2(1 - iC)} \right] \qquad [\text{ IX-27b }]$$

where $C > 0$ characterizes an "up-chirp" and $C < 0$ a "down-chirp" pulse.



A Gaussian pulse with a frequency "down-chirp"

Inverting the transform, we see that

$$U(\zeta,\tau) = \left[\frac{\tau_0^2}{\tau_0^2 + i\,2\,b\,\zeta(1 - iC)}\right]\exp\left\{-\frac{(1 - iC)\,\tau^2}{2\left[\tau_0^2 + i\,2\,b\,\zeta\,(1 - iC)\right]}\right\} \qquad \text{[IX-28a]}$$

or rationalizing

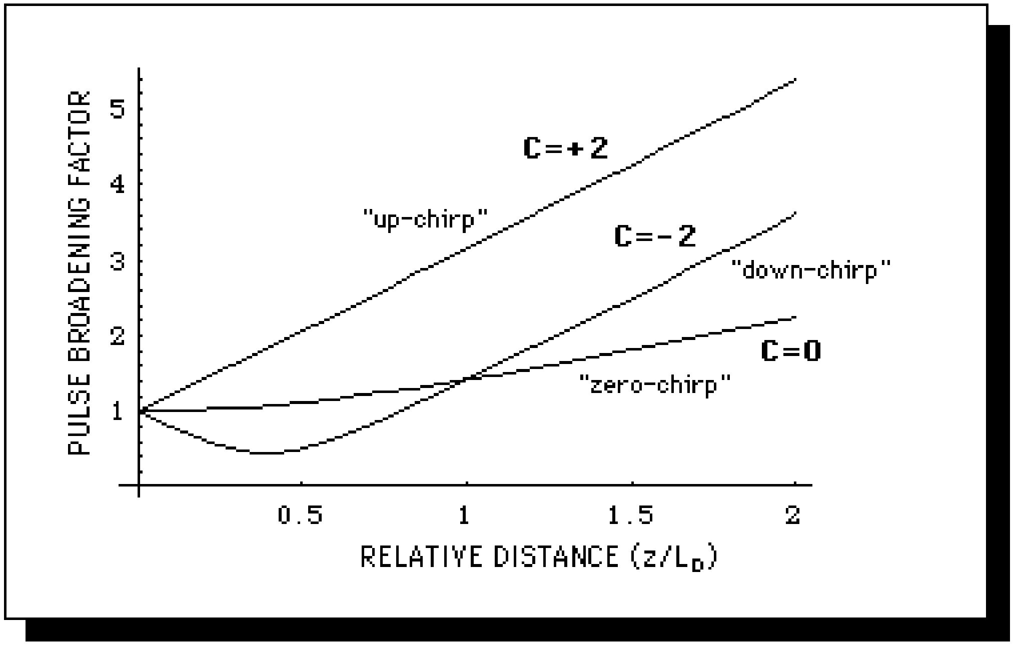
$$U(\zeta,\tau) = \left[\frac{\tau_0^2}{\tau_0^2 + i\,2\,b\,(1 - iC)}\right]\exp\left\{-\frac{\tau^2}{2\,\tau_0^2\left[\left(1 + \frac{2\,b\,C\,\zeta}{\tau_0^2}\right)^2 + \left(1 + \frac{2\,b\,\zeta}{\tau_0^2}\right)^2\right]}\right\}$$

$$\times \exp\left\{\frac{i\left[C + \frac{2\,b\,\zeta}{\tau_0^2}\left(1 + C^2\right)\right]\tau^2}{2\,\tau_0^2\left[\left(1 + \frac{2\,b\,C\,\zeta}{\tau_0^2}\right)^2 + \left(1 + \frac{2\,b\,\zeta}{\tau_0^2}\right)^2\right]}\right\} \qquad \text{[IX-28b]}$$

Hence, the **pulse width broadening factor** at a given position z is given by

$$\frac{\tau(\zeta)}{\tau_0} = \sqrt{\left(1 + \frac{2\,bC\,\zeta}{\tau_0^2}\right)^2 + \left(\frac{2\,b\,\zeta}{\tau_0^2}\right)^2} = \sqrt{\left(1 + \frac{C\,\zeta}{L_D}\right)^2 + \left(\frac{\zeta}{L_D}\right)^2} \qquad \text{[IX-29]}$$

where $L_D = \tau_0^2 / 2\,b$.



The spatial evolution of the pulse width of chirped Gaussian pulse

(For "normal dispersion" -- *i.e.* $\dfrac{\partial^2\beta}{\partial\omega^2} > 0$ **)**

<u>Pulse Solutions of “Negligible Dispersion” Nonlinear Schrödinger Equation</u>

If group velocity dispersion can be neglected, Equation [IX-17] reduces to

$$\frac{\partial \vec{\mathbf{G}}(\varsigma,\tau)}{\partial\varsigma} = -\frac{\alpha}{2}\,\vec{\mathbf{G}}(\varsigma,\tau) - i\,\gamma\left|\,\vec{\mathbf{G}}(\varsigma,\tau)\right|^2 \vec{\mathbf{G}}(\varsigma,\tau) \qquad \text{[IX-30]}$$

and if we take $\vec{\mathbf{G}}(\varsigma,\tau) \quad U(\varsigma,\tau)\sqrt{P_0}\,\exp(-\alpha\varsigma/2)$ we obtain

$$\frac{\partial U(\varsigma,\tau)}{\partial\varsigma} = -i\,\mathrm{L}_{\mathrm{NL}}^{-1}\,\exp(-\alpha\varsigma)\left|\,U(\varsigma,\tau)\right|^2 U(\varsigma,\tau) \qquad \text{[IX-31]}$$

where the characteristic nonlinear length is given by $\mathrm{L}_{\mathrm{NL}} = \left(\gamma\, P_0\right)^{-1}$. A solution to this equation is readily obtain in the form

$$U(\varsigma,\tau) = U(0,\tau)\ \exp\left[i\ \Phi_{\mathrm{NL}}(\varsigma,\tau)\right] \qquad \text{[IX-32a]}$$

where

$$\begin{aligned}\Phi_{\mathrm{NL}}(\varsigma,\tau) &= -\left[1-\exp(-\alpha\varsigma)\right]\left[\alpha\,\mathrm{L}_{\mathrm{NL}}\right]^{-1}\exp(-\alpha\varsigma)\left|U(0,\tau)\right|^2 \\ &= -\left[\varsigma_{\mathrm{eff}}/\mathrm{L}_{\mathrm{NL}}\right]\left|U(0,\tau)\right|^2\end{aligned} \qquad \text{[IV-32b]}$$

and

$$\varsigma_{\mathrm{eff}} = \left[1-\exp(-\alpha\varsigma)\right]\alpha^{-1} \qquad \text{[IX-32c]}$$

This interesting result show that so called “self-phase modulation” or SPM gives rise to an **intensity dependent phase shifted or chirped pulse which remains constant in shape** as it propagates. The instantaneous optical frequency shift is given by

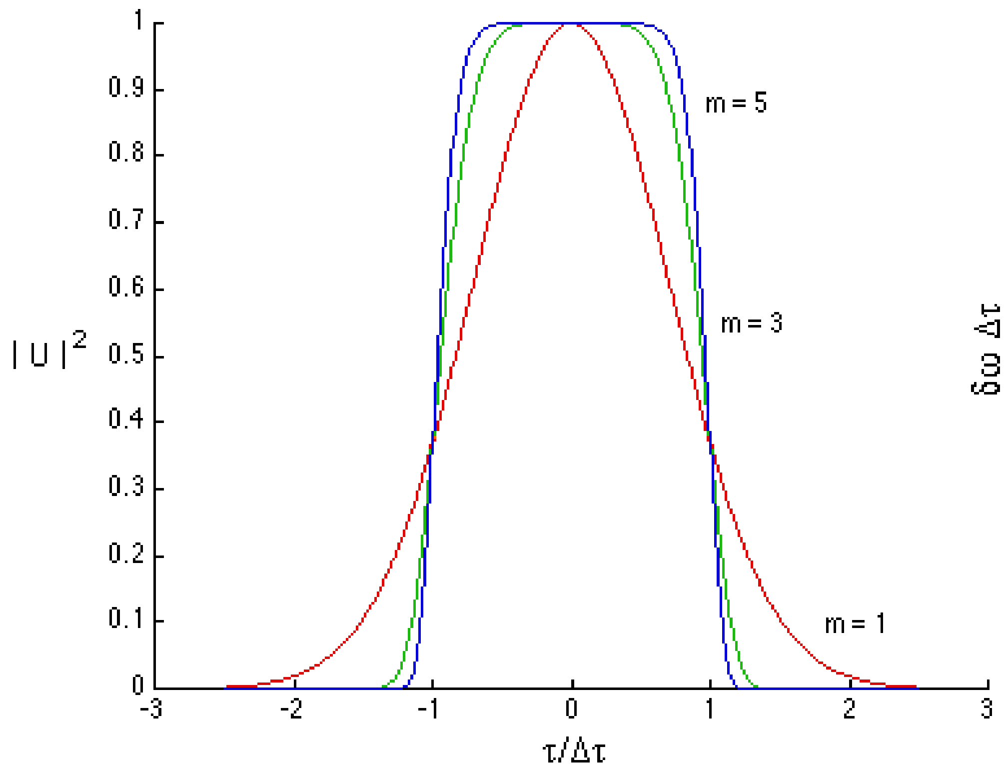
$$\delta\omega(\tau) = \frac{\partial\Phi_{\mathrm{NL}}(\varsigma,\tau)}{\partial\tau} = -\left[\varsigma_{\mathrm{eff}}/\mathrm{L}_{\mathrm{NL}}\right]\frac{\partial\left|U(0,\tau)\right|^2}{\partial\tau} \qquad \text{[IV-33]}$$

Note that the pulse spectrum is “red-shifted” on the leading edge of a pulse and “blue-shifted” on the trailing edge of the pulse. If we suppose the initial pulse to be a super-Gaussian of mth-order -- *i.e.*,

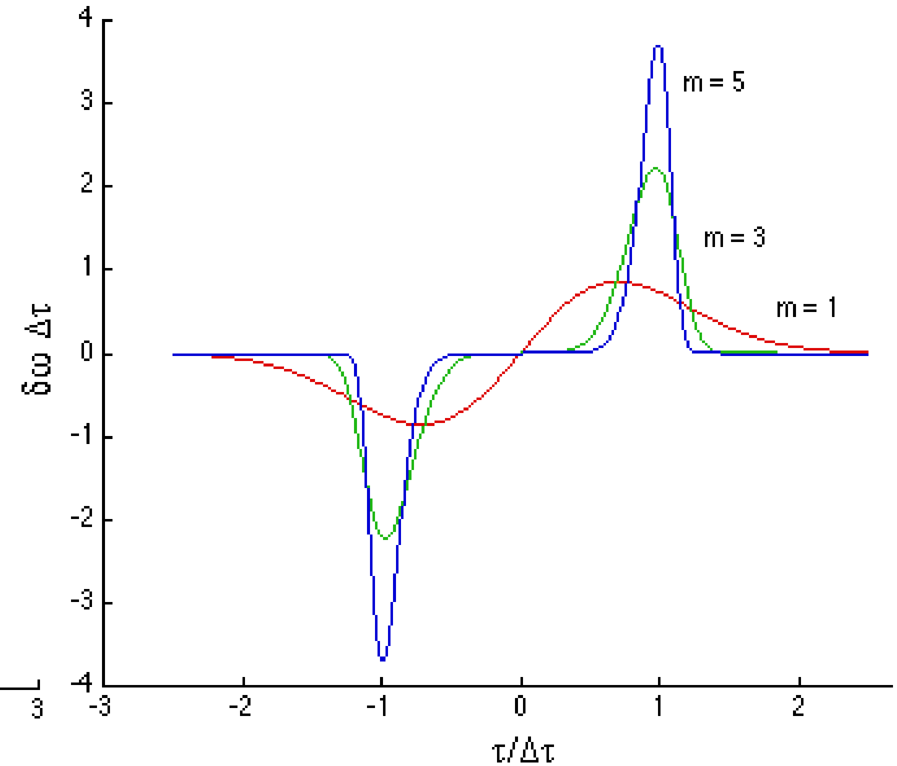
$$U_{\mathrm{m}}(0,\tau) = \exp\left[-\frac{1+iC}{2}\left(\frac{\tau}{\Delta\tau}\right)^{2\mathrm{m}}\right] \qquad \text{[IV-34]}$$

-- then the instantaneous optical frequency shift would be given by

$$\delta\omega(\tau) = \left(\frac{\varsigma_{\mathrm{eff}}}{\mathrm{L}_{\mathrm{NL}}}\right)\left(\frac{2\mathrm{m}}{\Delta\tau}\right)\left(\frac{\tau}{\Delta\tau}\right)^{2\mathrm{m}-1}\exp\left[-\left(\frac{\tau}{\Delta\tau}\right)^{2\mathrm{m}}\right] \qquad \text{[IV-35]}$$



Super-Gaussian Envelope Shapes



SPM Induced Frequency Chirp

A Brief Discussion of Solitons

We return briefly to Equation [IX-18] -- the standard form of the **nonlinear Schrödinger (NLS) equation** -- *i.e.*,

$$i\,\frac{\partial u(Z,T)}{\partial Z} - \operatorname{sgn}(\beta_2)\,\frac{\partial^2 u(Z,T)}{\partial T^2} + N^2\left|u(Z,T)\right|^2 u(Z,T) = 0 \qquad [\text{ IX-18 }]$$

where $u(Z,T) = \dfrac{\vec{\mathbf{G}}(\varsigma,\tau)}{\sqrt{P_0}}$, $Z = \dfrac{\varsigma}{L_D} = \varsigma \dfrac{|\beta_2|}{\Delta\tau^2}$, $T = \sqrt{2}\,\dfrac{\tau}{\Delta\tau}$ and $N^2 = L_D / L_{NL} = \gamma P_0 \Delta\tau^2 / |\beta_2|$.

In soliton analysis the most common form of NLS equation is the following:

$$i\frac{\partial U(Z,T)}{\partial Z} + \frac{\partial^2 U(Z,T)}{\partial T^2} + |U(Z,T)|^2 U(Z,T) = 0 \qquad \text{[IX-36]}$$

where $U(Z,T) = N\ u(Z,T) \quad \vec{\mathbf{G}}(\varsigma,\tau)\sqrt{\dfrac{\gamma\,\Delta\tau^2}{|\beta_2|}}$.

If $N \geq 1$ the following ***fundamental soliton*** will propagate undistorted for an arbitrary distance:

$$U(Z,T) = \operatorname{sech}(T)\ \exp(i\,Z/2) \qquad \text{[IX-37]}$$

If $N \geq 2$ the following ***second-order soliton*** will propagate undistorted for an arbitrary distance:

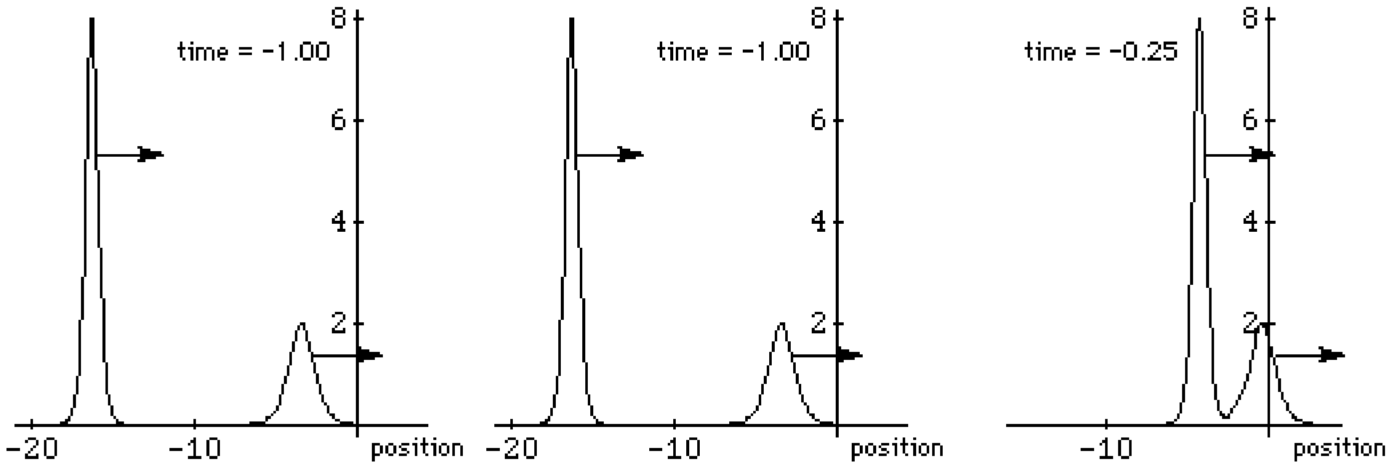
$$U(Z,T) = \frac{4\left[\cosh(3\,T) + 3\exp(i\,4\,Z)\cosh(T)\right]}{\left[\cosh(4\,T) + 4\cosh(2\,T) + 3\cos(4\,Z)\right]}\ \exp(i\,Z/2) \qquad \text{[IX-37]}$$

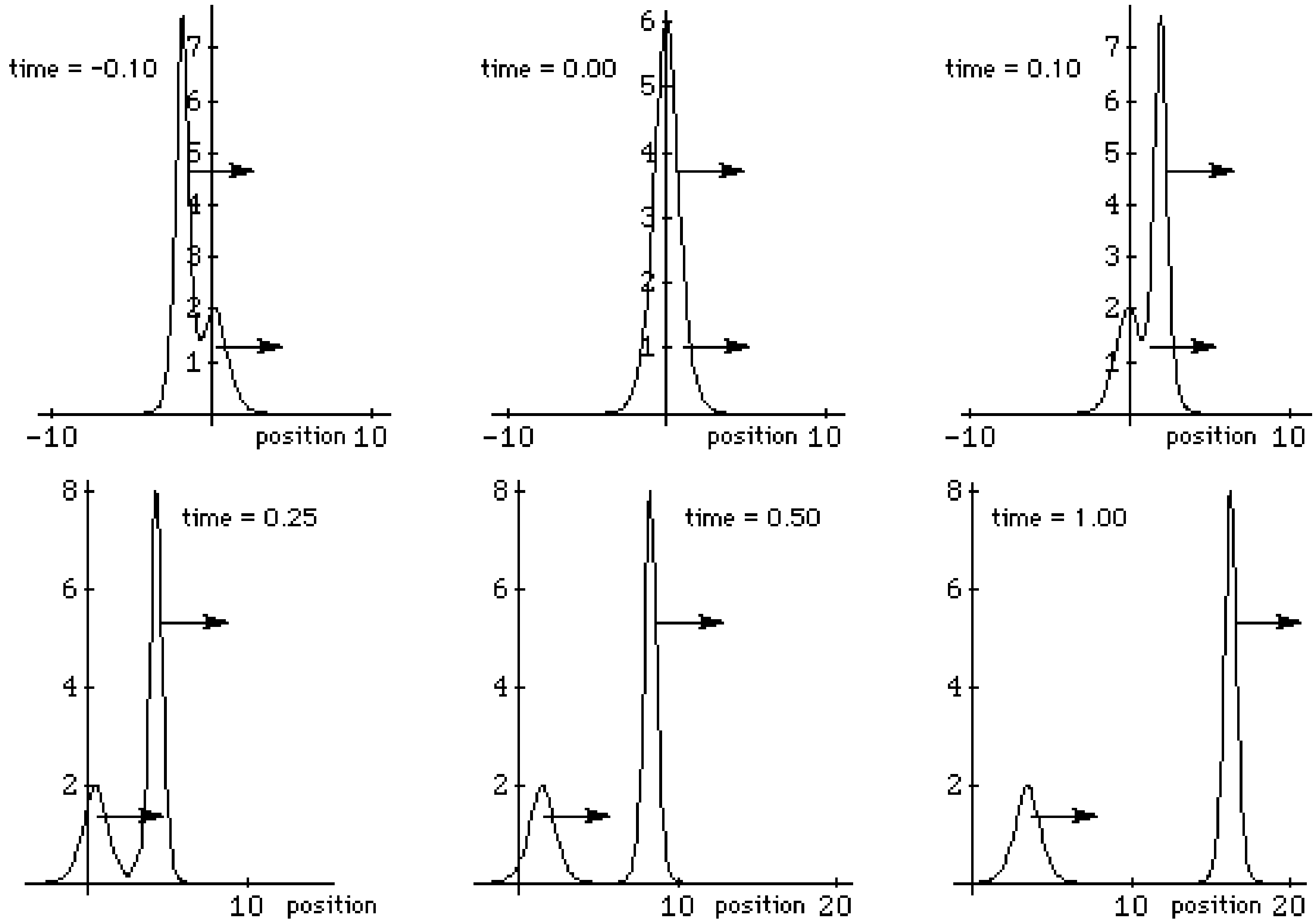
$$|U(Z,T)|^2 = \frac{16\left[\cosh^2(3\,T) + 9\cosh^2(T) + 6\cos(4\,Z)\cosh(3\,T)\cosh(T)\right]}{\left[\cosh(4\,T) + 4\cosh(2\,T) + 3\cos(4\,Z)\right]^2} \text{ [IX-37]}$$

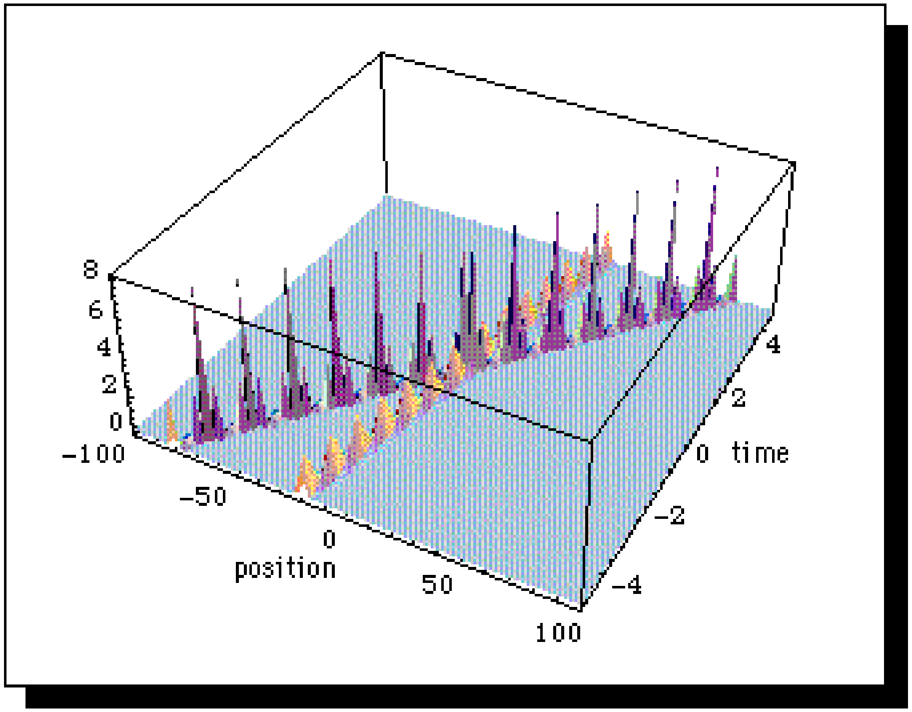
TWO SOLITON COLLISION ILLUSTRATIONS

The following are time-lapsed illustration of the propagation and collision of two solitons -- plot here is

$$u(x,t) = 12\,\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{\left[3\cosh(x - 28t) + \cosh(3x - 36t)\right]^2}$$




time = -0.10
position
time = 0.00
time = 0.10
time = 0.25
time = 0.50
time = 1.00



8
6
4
2
0
-100
-50
0
position
50
100
4
2
0
time
-2
-4